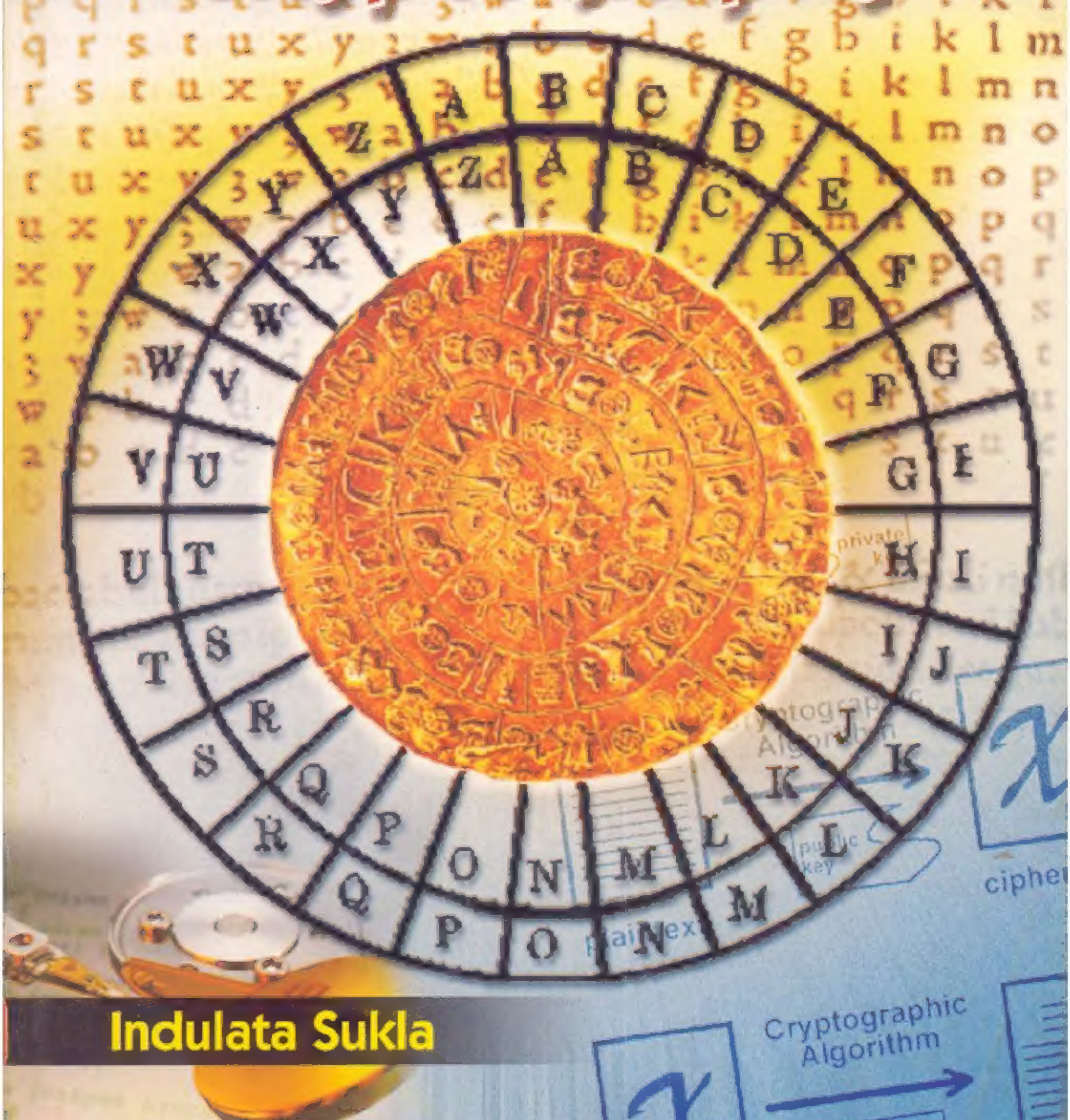
NUMBER THEORY AND ITS APPLICATION TO CRYPTOGRAPHY



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NUMBER THEORY AND ITS APPLICATION TO CRYPTOGRAPHY

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In the sacred memory of my late parents

Kashinath Kar Mohapatra and Swarna Mayee Devi

PREFACE

The present book is evolved from a course on analytic number theory offered at the Mathematics department of Sambalpur University during the last 25 years. It provides the basic knowledge of number theory and its application to cryptography suitable for one semester course in the Post Graduate level and also serves as a reference book for the teachers and for the beginners in number theory. The topics chosen and illustrated in this book are intended to provide some depth in the subject. Many problems of much interest are discussed together with some more exercises left for the students. The computer programming in BASIC are given at some places which will help students to work out in the computer for practice. In some cases algorithms are also described.

The goal of this work is to attract students to opt for the subject and do some fruitful researches in this field. Many of the students who have taken this course during the past 25 years have become professional teachers and contributed to number theory.

Apart from my students who have taken keen interest in my lectures, my sons Chuni and Lulu have always encouraged me in writing this book

Lastly I acknowledge the authority of Sambalpur university for sanctioning me sabbatical leave to write this book.

Makara Sankranti, 2004 **INDULATA SUKLA**

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CHAPTER 0

INTRODUCTION

The theory of numbers is that branch of mathematics which deals with properties of whole numbers, 1, 2, 3, 4, 5…. that is positive integers. Numbers, specially positive integers were used for keeping records and for commercial transaction for over 5000 years before any one thought of studying numbers in a systematic way. Many branches of mathematics have their origin in number theory. the first scientific approach to the study of integers that, the true origin of the theory of numbers, is generally attributed to the Greeks. Around 600 BC Pythagoras and his disciple made rather thorough studies of integers. They classified them to four categories even integers, odd integers, prime numbers and composite numbers.

A prime number is a number greater than 1 whose only divisors are 1 and the number it self. Numbers which are not prime are called as composite number. But the number 1 is neither a prime nor a composite number.

Pythagoreans also linked numbers with geometry. When the numbers are represented by dots arranged in the form of triangles, squares, pentagons, haxagons, septagons. octagons etc. they are called as triangular numbers, square numbers pentagonal numbers, septagonal number, octagonal numbers respectively . In general if a number is represented by dots arranged in form of a polygon then the number is called polygonal number. These numbers specially triangular numbers and pentagonal numbers have importance in the study of partition theory.

Around 300 BC Euclid's *Elements* a collection of 13 books appeared. Three of these 13 books were devoted to the theory of numbers. (BOOK VI, IX and X). Euclid was the first mathematician to prove that there are infinite number of primes. There are also proofs of this theorem by other eminent mathematicians. There is also topological proof of this famous Euclid's theorem. Also we note that some primes are represented by the numbers of the form 4k ± 1, 6k ± 5, 8k ± 3, 8k ± 1, k = 1, 2, 3, ... Like Euclids theorem, we have theorems which

The prime number theorem tells that

$$\lim_{x \to \infty} \frac{\pi(x) \log x}{x} = 1.$$

This conjecture was made independently by Gauss and Legendre. They attempted to prove this statement but failed. Later on in 1851 the Russian mathematician Chebyshev gave equivalent theorems of this statement.

Besides prime numbers there are other numbers in the world of numbers which have interesting properties. They are amicable numbers, Armstrong numbers and Fibonacci numbers.

Two integers m and n are said to be amicable or friendly numbers if the sum of the proper divisors of m is the number n and vice versa. The smallest amicable number pair is 220 and 284 because the divisors of 220 are 1, 2, 34, 5. 10, 11, 20, 22, 44, 55, 110 when added equals to 284 and again the divisors of 284 are 1, 2, 4, 71, 142, when added gives 220. There are other amicable pairs also. If the sum of the divisors of an integer n is less than 2n, then it is called as 'deficient number'. On the other hand if the sum of the divisors of integer n is greater than 2n, then n is called as abundant number. A sequence of integers u_1, u_2, . . . u_n, is called Fibonacci sequence if $u_1 = 1$, $u_2 = 1$, $u_{n+1} = u_n + u_{n-1}$.

There are many interesting properties of Fibonacci numbers.. It is applied in Golden ratio.

The most emerging field of modern time is the Cryptography, the science of making communication unintelligible to all except authorized parties. Cryptography is the only known practical means for protecting information transmitted through public communication net works. It is a subject of common interest to both mathematician and computer scientists. Number theory has application to Crypography. In 1977, R. Rivest, A. Shamir, and L. Adleman proposed a public key cryptosystem which uses only elementary ideas from number theory. Their system is popularly known as RSA after the initials of the inventors. Its security depends on the assumption that the factorization of composite numbers with large prime factors is time taking even in the computer. In this system each user chooses a pair of distinct primes, p and q, so large that the factorization of their product n = pq, is beyond all computational capabilities. After selecting n, the user chooses a random positive integer k, and the pair (n, k) is publicly known. Because the factors p and q are not known easily it is not possible for out siders party to decipher the message thus the security is maintained. Also it is not possible to say whether a given large number n is a prime or composite. It has to go through different primality tests. If n is not prime but composite there are methods due to Fermat and others to find factors of the large number n. There are other different methods for the primality testing. It is interesting to study these methods.

Another application of congruential arithmetic is a public-key encryption method in which message represented by integers are raised to a given power and only the residues modulo a preselected encryption modulus is transmitted from sender to receiver.

For example if the message is 7, the encryption key is 3 and the modulus is 10, then the transmitted message would not be $7^3 = 343$, but only the digit : 3. Chapter 5 and chapter 8 are devoted to the topic cryptography and primality and factoring, which contains the application of Number theory to cryptography.

◆◆◆

CHAPTER 1

PRIME NUMBERS AND DISTRIBUTION OF PRIME NUMBER

1.1. PRIME NUMBERS.

The numbers 1, 2, 3, ... are called positive integers. Among the positive integers there is as subclass of peculiar importance, the class of primes. A number p is said to be prime if

(*i*) $p > 1$

(*ii*) p has no positive divisor except 1 and p.

NOTE An integer a is said to be divisible by another integer b, not o, if there is a third integer c such that

$$a = b\,c.$$

In this case b is said to be a positive proper divisor of a.

A number greater than 1 but not prime is called as composite .

Example : 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 47 are the primes less than 50.

Now the question is how to find out the primes. The process is known as primality test . The first elementary test is sieve method

SIEVE OF ERATOSTHENES : In order to find prime number between 1 and 100 we proceed as follows:

STEP 1 : Write the numbers 1 to 100 in tabular form from 1 to 6 in column 1, 7 to 12 in column 2 and so on then proceed step by step

STEP 2 : Consider the second row starting with 2 strike out all multiples of 2 in the table and round 2.

STEP 3 : Consider third row starting with 3 and strike out all multiples of 3 from the table and round 3.

STEP 4 : Take 5th row starting with 5. Strike out all multiples of 5 from the rest of numbers in the table excepts 5. Round 5.

STEP 5 : Round the numbers left out in the table after following step 2 to step 4.

These are the numbers which are prime numbers between 1 and 100. 1 is not considered as a prime number though it is divisible by 1 and so there are 25 primes between 1 to 100. If you are asked to find out primes between 100 to 200 the same procedure is repeated Here one has to consider the multiples of 7, 11 and so on. This process is most elementary and can not be tried with a very big number with 10 million digits.

The following BASIC programme computes all the prime numbers upto 1000 using the sieve of Erastothenes.

```
100        REM SIEVE OF ERATOSTHENES
110        DIM N [1000], P[200]
120        FOR  1 = 2 TO 1000
130        LET N [I] = 0
140        NEXT I
150        LET K = 0
160        FOR P = 2 TO 1000
170        IF N[P] < 0 THEN 240
180        LET K = K + 1
190        LET P[K] = P
200        IF P > SQR (1000) THEN 240
210        FOR I = P TO 1000 STEP P
220        LET N[I] = –1
230        NEXT I
240        NEXT P
250        REM PRINT PRIME NUMBERS
260        LET C = 1
270        FOR I = 1 TP K
280        PRINT P[I];
290        LET C = C+1
```

```
300            IF C < = 7 THEN 330
310            PRINT
320            LET C = 1
330            NEXT I
340            END
```

RUN

```
  2    3    5    7   11   13   17   19   23
 29   31   37   41   43   47   53   59   61
 67   71   73   79   83   89   97  101  103
107  109  113  127  131  137  139  149  151
157  163  167  173  179  181  191  193  197
199  211  223  227  229  233  239  241  251
257  263  269  271  277  281  283  293  307
311  313  317  331  337  347  349  353  359
367  373  379  383  389  397  401  409  419
421  431  433  439  443  449  457  461  463
467  479  487  491  499  503  509  521  523
541  547  557  563  569  571  577  587  593
599  601  607  613  617  619  631  641  643
647  653  659  661  673  677  683  691  701
709  719  727  733  739  743  751  757  761
769  773  787  797  809  811  821  823  827
 29  839  853  857  859  863  877  881  883
887  907  911  919  929  937  941  947  953
967  971  977  983  991  997
```

THEOREM 1 : Every positive integer except 1, is a product of primes.

PROOF : Either n is a prime, when there is nothing to prove; or n has divisors between 1 and n. If n is composite then there exists an integer d satisfying $d \mid n$ and $1< d <n$. Among all such integers d choose p_1 to be the smallest. Then p_1 must be a prime number otherwise it would have a divisor q with $1 < q < p_1$, but then $q \mid p_1$, and $p_1 \mid n$ imply $q \mid n$ which contradicts the choice of p_1 as the smallest divisor not equal to 1.

We write $n = p_1n_1$, where p_1 is a prime and $1 < n_1 < n$. If n_1 is prime we are done. Otherwise the process is repeated to find another prime p_2 such that $n_1 = p_2n_2$, that is

$$n = p_1p_2n_2 \qquad 1< n_2 < n_1.$$

Repeating the argument we have now

(1) $n = p_1p_2p_3 \ldots .p_k.$

1.2. GREATEST COMMON DIVISOR :

If d divides two integers a and b, then d is called a common divisor of a and b.

THEOREM 2 : Given any two integers a and b, there is a common divisor d of a and b of the form

(2) $d = ax + by$

where x and y are integers. Moreover, every common divisor of a and b divides this d.

PROOF : First we assume that $a \geq 0$ and $b \geq 0$.

We use induction on n; where $n = a + b$. If $n = 0$ then $a = b = 0$ and we take $d = 0$ with $x = y = 0$. Assume the theorem is true for $0, 1, 2, \ldots . n - 1$. To prove it for n, assume $a \geq b$. If $b = 0$ take $d = a$, $x = 1$, $y = 0$. If $b \geq 1$ apply the theorem to $a - b$ and b. Since $(a - b) + b = a = n - b \leq n - 1$, by induction hypothesis there is a common divisor d of $(a - b)$ and b of the form $d - (a - b)x + by$. This d also divides $a - b + b = a$ so d is a common divisor of a and b, and we have $d = ax + (y - x)b$, a linear combination of a and b. Since a common divisor divides a and b and hence by linearity it divides d. Hence proved.

If $a < 0$ or $b < 0$ (or both), we can apply the result to $|a|$ and $|b|$. Then there is a common divisor d of $|a|$ and $|b|$ of the form

$$d = |a|\, x + |b|\, y$$

If $a < 0$, $|a|\, x = - a\, x = a\, (-x)$. Similarly $b < 0$, $b|\, y = b\, (-y)$. Hence d is again a linear combination of a and b.

Definition : (Greatest common divisor)

Given integers a and b, there is one and only one number d with the following properties :

(*a*) $d \geq 0$

(*b*) $d \mid a$ and $d \mid b$

(*c*) $e \mid a$ and $e \mid b$ implies $e \mid d$.

Then d is called as the greatest common divisor (gcd) of a and b and is denoted by (a,b). gcd (–10, 34) = 2. If d = (a,b) = 1 then a and b are said to be relatively prime.

By theorem 2, there exists integers x and y such that 1 = ax + by.

Corollary 1 : If gcd (a, b) = d, then gcd (a/d, b/d) = 1.

PROOF : Since gcd (a, b) = d it is possible to find integers x and y such that d = ax + by. Dividing each side of this equation by d, we get the expression

$$1 = (a/d)\, x + (b/d)\, y.$$

Because a/d and b/d are integers then (a/d, b/d) = 1, i.e. a/d and b/d are relatively prime.

Corollary 2 : If $a \mid c$ and $b \mid c$, with gcd (a, b) = 1, then $ab \mid c$.

PROOF : As $a \mid c$ and $b \mid c$ we can find integers r and s such that c = a r = b s. Now gcd (a,b) = 1 allows us to write 1 = ax + by for some integers x and y. Multiplying by c we get

$$c = c.1 = c\,(ax + by) = acx + bcy.$$

Substituting values of c on the right hand side, we get

$$c = a\,(bs)\,x + b\,(ar)\,y = ab\,(sx + ry)$$

which implies that $ab \mid c$.

THEOREM 3 : (Euclid's lemma) If $a \mid bc$ and if (a,b) = 1, the $a \mid c$.

PROOF : Since (a,b) = 1, we can write by theorem 2, 1 = ax + by. Therefore c = acx + bcy. But $a \mid a\,c\,x$ and $a \mid b\,c\,y$, so $a \mid a\,c\,x + b\,c\,y = c$. Hence proved.

Remark: gcd (a,b) = 1 is a necessary condition for theorem 3. For Example $15 \mid 6\,.\,10$ but $15 \nmid 6$ and $15 \nmid 10$.

THEOREM 4 : If p is a prime and $p \mid ab$ then $p \mid a$ or $p \mid b$.

PROOF : If $p \mid a$ then we need not go further. So let us assume that $p \mid a$. Since the only positive divisors of p are 1 and p itself, this implies that gcd (p,a) = 1. But g cd (p,a) = p or gcd (p,a) = 1 according as $p \mid a$ or $p \mid a$.Since $p \mid ab$ and $p \mid a$ by Euclid lemma we get $p \mid b$.

Corollary 3. If p is a prime and $p \mid a_1, a_2 \ldots\ldots a_n$ then $p\mid a_k$ for some k, where $1 \leq k < n$.

PROOF : We proceed by induction on n, the number of factors. When n = 1, conclusion is trivial. When n = 2 the result is theorem 4. Assume $n \geq 2$ and that whenever p divides a product less than n factors, then it divides at least one of its factors. Now let $p \mid a_1a_2\ldots.a_{n-1}$. According to theorem 1 $p \mid a_n$ or $p\mid a_1, a_2 \ldots a_{n-1}$. If $p \mid a_n$ then we are done. For the case $p\mid a_1a_2 \ldots . a_{n-1}$ by induction hypothesis $p \mid a_k$ for k with $1 \leq k \leq n-1$. Hence p divides one of the integers $a_1, a_2 \ldots a_n$.

Corollary 4. If $p, q_1, q_2 \ldots\ldots q_n$ are all primes and $p \mid q_1, q_2 \ldots\ldots\ldots q_n$ then $p \mid q_k$ for some k, where $1 \leq k < n$.

PROOF : By corollary 1, we know that $p \mid q_k$ for some k, with $1 \leq k \leq n$. Being a prime , q_k is not divisible by any positive integer other than 1 or q_k itself. Since p > 1, we conclude that $p = q_k$.

THEOREM 5 : (Fundamental Theorem of Arithmetic)

The representation of every integer n > 1 as a product of primes in (1) is unique apart from the order in which the factors occur.

PROOF : Let us suppose that the integer n can be represented as a product of primes in two ways say

$$n = p_1p_2p_3 \ldots .p_{r\cdot} = q_1q_2q_3 \ldots .q_s,$$

where the p_j and q_j are all primes, written in increasing magnitude so that

$$p_1 \leq p_2 < \ldots.. \leq p_r,\ q_1 \leq q_2 \leq \ldots.. \leq q_s.$$

Since $p_1 \mid q_1q_2 \ldots . q_s$ then $p_1 = q_k$ for some k, but then $p_1 \geq q_1$. Similar reasoning gives $q_1 \geq p_1$, whence $p_1 = q_1$. We may cancel this common factor and obtain

$$p_2p_3 \ldots .p_{r\cdot} = q_2q_3 \ldots .q_s.$$

Continuing we get if r < s

$$1 = q_{r+1}\, q_{r+2} \ldots . q_s$$

which is absurd, since each $q_i > 1$ hence r = s and

$$p_1 = q_1, p_2 = q_2 \ldots\ldots.. p_r = q_r$$

giving the uniqueness of the two representations of n.

Any positive integer n > 1 has canonical form

$$n = p_1^{\alpha 1} p_2^{\alpha 2} \ldots p_n^{\alpha n}$$

(3) or $n = \prod_{i=1}^{r} p_i^{\alpha_i}$, where $p_1 \leq p_2 \leq \ldots.. \leq p_r \, \alpha_i'. \, s \geq 0$.

This is called as the standard form of n.

THEOREM 6 : If $n = \prod_{i=1}^{r} p_i^{a_i}$, the set of positive divisors of n is the set of numbers of the form $\prod_{i=1}^{r} p_i^{ci}$, where $0 < c_i \leq a_i$, for $i = 1, 2, \ldots r$.

PROOF : For $0 \leq c_i \leq a_i$ $p_i^{c_i}$ divides $p_i^{a_i}$.

So $\prod_{i=1}^{r} p_i^{c_i}$ divides $\prod_{i=1}^{r} p_i^{a_i}$.

THEOREM 7 : If two positive integers a and b have the factorization

$a = \prod_{i=1}^{\infty} p^{a_i}$, $b = \prod_{i=1}^{\infty} p_i^{bi}$, then their gcd has the factorization

$$(a, b) = \prod_{i=1}^{\infty} p_i^{c_i},$$

where $c_i = \min \{a_i, b_i\}$,

PROOF : Let $d = \prod_{i=1}^{\infty} p_i^{c_i}$. Since $c_i \leq a_i$ and $c_i \leq b_i$ we have d divides a and d divides b so d is a common divisor of a and b. Let e be a common divisor a and b, and write e

$$e = \prod_{i=1}^{\infty} p_i^{e_i}. \text{ Then } e_i \leq a_i \text{ and } e_i \leq b_i, \text{ so } e_i \leq c_i,$$

Hence $e \mid d$. So d is the gcd of a and b.

Euclidean Algorithm

We describe the Euclidean Algorithm as follows : Let a and b be two integers whose greatest common divisor is desired. Since gcd $(|a|, |b|) =$ gcd (a,b) there is no harm in assuming that $a \geq b > 0$. By division algorithm we find integers q_1 and r_1 such that

$$a = q_1 b + r_1 \qquad 0 \leq r_1 < b.$$

If it happens that $r_1 = 0$ then b |a and gcd (a,b) = b when $r_1 \neq o$, divide b by r_1 to produce integers q_2 and r_2 satisfying

$$b = q_2 r_1 + r_2 , \qquad 0 \leq r_2 < r_1.$$

If $r_2 = 0$, then we stop; otherwise, proceeding as before we obtain

$$r_1 = q_3 r_2 + r_3 \qquad 0 \leq r_3 < r_2.$$

This division process continues until some zero remainder appears, say at the (n+1) the stage where r_{n-1} is divided by r_n.

We get a decreasing sequence $b > r_1 > r_2 > \ldots\ldots . \geq 0$ satisfying the following system of equation :

$$\begin{aligned} a &= q_1 b + r_1, && 0 \leq r_1 < b \\ b &= q_2 r_1 + r_2, && 0 < r_2 < r_1 \\ r_1 &= q_3 r_2 + r_3, && 0 < r_3 < r_2 \\ &. \\ &. \\ r_n &= q_n r_{n-1} + r_n, && 0 < r_{n-2} < r_{n-1} \\ r_{n-1} &= q_{n+1} r_n + 0 && 0 < r_n < r_{n-1}. \end{aligned}$$

Now r_n = gcd (a,b).

Since if d = gcd (a, b) then d | a and d | b imply that d | ($a-q_1b$) or d|r_1 by the relation $a = qb + r_1$. Thus d is a common divisor of both b and r_1. if c is a common divisor of b and r_1 then c| $q_1b + r_1 = a$. So c is a common divisor of both a and b . Hence $c \leq d$. It now follows from the definition of gcd (b, r_1) that d = gcd (a,b) gcd (b_1 r_1) = gcd (r_1 , r_2) = ….. = gcd ($r_{n,}$ 0) = r_n. hence r_n is the greatest common divisor of integers a and b. We can express r_n as a linear combination of a and b.

Example 1. Find gcd (1998, 2000).

$$2000 = 1998 \times 1 + 2$$

$$1998 = 2 \times 999 + 0$$

Hence gcd (1998, 2000) = 2.

We express 2 as a linear combination of 1998 and 2000.

$$2 = 2000 \times 1 + 1998 (-1).$$

Example 2. Find gcd (143, 227).

$$227 = 143 \times 1 + 84$$

$$143 = 84 \times 1 + 59$$

$$84 = 59 \times 1 + 25$$
$$59 = 25 \times 2 + 9$$
$$25 = 9 \times 2 + 7$$
$$9 = 7 \times 1 + 2$$
$$7 = 2 \times 3 + 1$$
$$2 = 1 \times 2 + 0$$

Hence gcd (143, 227) = 1.

We express 1 as a linear combination of 143 and 227.

$$1 = 7 - 6$$
$$= 7 - 2 \times 3$$
$$= 7 - 3 (9 - 7)$$
$$= 4 \times 7 - 3 \times 9$$
$$= 4 \times (25 - 9 \times 2) - 3 \times 9$$
$$= 4 \times 25 - 11 \times 9$$
$$= 4 \times 25 - 11 (59 - 25 \times 2)$$
$$= 26 \times 25 - 11 \times 59$$
$$= 26 \times (84 - 59) - 11 \times 59$$
$$= 26 \times 84 - 37 \times 59$$
$$= 26 \times 84 - 37 \times (143 - 84)$$
$$= 63 \times 84 - 37 \times 143$$
$$= 63 \times (227 - 143) - 37 \times 143$$
$$= (6 \times 3) 227 - (100) 143.$$

The gcd of 76084 and 63020 is 92.

The following BASIC program uses Euclid's algorithm to compute the gcd of given pair of numbers.

```
100    REM FIND THE GCD  OF TWO NUMBERS
120    PRINT"    A","    B"," GCD "
130    PRINT
140    READ A,B
150    PRINT A,B
160    LET Q = INT (A/B)
170    LET R = A –Q * B
180    LET A = B
```

```
190    LET B = R
200    IF R > 0 THEN 160
210    PRINT A
220    GPTP 140
230    DATA 60, 5280, 49, 139, 3850, 5280
240    DATA 1124, 1472, 17296, 18416
250    DATA 76084, 63020, 7854, 13398
260    END
```

RUN

```
   A      B GCD :     60   5280     60,    49    139     1,
 3850   5280   110,   1124   1472     4,  17296  18416    16
76084  63020     92, 7854    13398   462.
     OUT OF DATA IN LINE 140.
```

1.3. HOW MANY PRIME NUMBERS ARE THERE ?

THEOREM 8 : If n is a natural number > 2 then between n and n! there is at least one prime number.

PROOF : As n>2, the integer $N = n! - 1$ is > 1 and, it has a prime divisor p, which is obviously $\leq N$ and so $< n$!. Now we can not have $p \leq n$, because then p would be a divisor of the number n!, and being also a divisor of the number N, it would be a divisor of the differences of these numbers, i.e. the numbers $n! - N = 1$ which is impossible. Therefore $p > n$ and as we know, $p < n!$ we have $n < p < n!$ and the theorem is proved .

In connection of the above theorem we observe that in 1850 Chebyshev proved a stronger theorem called as Bertrand postulate:

Between n and 2n there exists at least one prime p. That is $n \leq p < 2n$.

Therefore, for every natural number there exists a prime number greater than that number, whence it follows that there are infinite number of primes which was known to Euclid.

THEOREM 9 : (Euclid) There are infinite number of primes.

PROOF : Suppose that there are only a finite number, say $p_1\ p_2 \cdots p_n$. Let $N = 1 + p_1p_2 \ldots p_n$. Now $N > 1$, so either N is prime or N is a product of primes. N is not prime since it exceeds each p_i . Moreover no p_i divides N. Because if p_i divides N, then p_i divides the difference $N - p_1p_2 \ldots\ p_n = 1$. This is a contradiction. This proves the theorem .

Second proof of Euclid's theorem: Suppose that 2, 3 …p_j are the first j primes and let N(x) be the number of n not exceeding x which are not divisible by any prime $p > p_j$. If we express such an n in the form

$$n = n_1{}^2 m ,$$

where m is 'quadrifrei' i.e. is not divisible by the square of any prime, we have

$$m = 2^{b1}\, 3^{b2} \ldots . \; p_j{}^{bj}$$

with every b either 0 or 1. There are just 2j possible choices of the exponents and so not more than 2^j different values of m. Again $n_1 < \sqrt{n} \le \sqrt{x}$ and so there are not more than $\sqrt{x}$ different values of n_1. Hence

(4) $N(x) \le 2^j \sqrt{x}$. .

If the theorem is false, so that the number of primes is finite, let the primes bc 2,3. . . .p_j. In the case N (x) = x for every x and so $x \le 2^j \sqrt{x}$ implies $x \le 2^{2j}$ which is false for $x \ge 2^{2j} + 1$.

THEOREM 10 : The infinite series $\sum\limits_{n=1}^{\infty} 1/p_n$ is divergent .

PROOF : We will use the above arguement for the proof of this theorem. If the series is convergent, we can choose j, so that the remainder after j terms is less than ½ i.e.

(5) $\dfrac{1}{p_{j+1}} + \dfrac{1}{p_{j+2}} = - - - - - - - < \dfrac{1}{2}$ The number $n \le x$ which are divisible by p are atmost x / p . Hence x--N (x), the number of $n \le x$ divisible by one or more $p_{j+1}, p_{j+2} \ldots$ is not more than

$$\frac{1}{p_{j+1}} + \frac{1}{p_{j+2}} = - - - - - - - < \frac{1}{2}x$$

Hence by (4)

$$½\, x < N(x) < 2^j \sqrt{x}$$

i.e.

$x < 2^{2j+2}$, which is false for $x \ge 2^{2j+2}$.

Hence the series diverges.

THEOREM 11 : There are infinitely many primes of the form 4n–1.

PROOF : Define N by $N=2^2.3.5\ldots p-1$. Then N is of the form 4n–1 and is not divisible by any of the primes upto p. It can not be a product of primes of the form 4n+1 only, since the product of two numbers of this form is of the same form; and therefore it is divisible by a prime 4n–1 greater than p. Hence there are infinitely many primes of the from 4n--1.

THEOREM 12 : There are infinitely many primes of the form 4n+3.

PROOF Define N by $N=2^2.3.5\ldots p-1$ Then N is of the form 4n+3, and is not divisible by any of the primes upto p. It can not be a product of primes 4n+1 only, since the product of two numbers of this form is of the form 4n+1; and therefore it is divisible by a prime 4n+3, greater than p . Hence there are infinitely many primes of the form 4n+3.

THEOREM 13 : There are infinitely many primes of the form 6n + 5.

PROOF : The proof is similar. Define N as $N=2.3.5\ldots p-1$.

We observe that any prime numbers, except 2 or 3, is 6n –1 or 6n + 5 and the product of two numbers of the form 6n+1 is the of the same form. Hence prime divisors of N are of the form 6n + 5 greater than p. Hence there are infinitely many primes of the form 6n+5.

THEOREM 14 : There are infinitely many primes of the form 8n + 5.

PROOF : We take $N = 3^2, 5^2, 7^2 \ldots p^2 + 2^2$, a sum of two squares which have no common factor . The square of an odd number 2m+1 is 4m(m+1) +1 and is 8n+1. So that N is 8n+5. By above theorem any prime factor of N is 4n+1, and so 8n+1 or 8n + 5 and that the product of two numbers 8n + 1 is of the same form, we complete the proof as before.

THEOREM 15 : If p_n is the nth prime number, then $p_n \le 2^{2^{n-1}}$.

PROOF : Let us proceed, by induction on n. For n = 1, it is trivial . Assume n > 1 and that the result holds for integers up to n. Then

$$\begin{aligned} p_{n+1} &\le p_1 p_2 \ldots p_n + 1 \\ &\le 2.2^2 \ldots 2^{2^{n-1}} + 1 = 2^{1+2+\ldots+2^{n-1}}+1 \\ &\le 2^{2^n-1} + 1 \end{aligned}$$

But $2^{2^n-1} \ge 1$for all n; whence

$$p_{n+1} \le 2^{2^n-1} + 2^{2^n-1} = 2.\ 2^{2^n-1} = 2^{2^n}$$

Hence proved.

Corollary : For n ≥ 1, there are at least n+1 primes less than 2^{2^n}.

PROOF : From the theorem we know that p_1 , p_2 ,. . p_n are all less than 2^{2^n}.

There exist prime numbers having at least three thousand digits, but no such number is known. The greatest known prime number has 1332 digits it is the number $2^{4423}-1$ which was verified to be prime in the year 1961. Most recently largest known prime is $2^{6,972,593}-1$ having 2, 098, 960 digits found by Nayan Hajaratwala in June 1999.

From Bertrands postulate it follows that for every natural number there exist at least three prime numbers each having s digits. Since each of the numbers

10^{s-1} ,2.10^{s-1} , 4.10^{s-1} and $8.\ 10^{s-1}$

have s digits, then for s>1 there exist prime numbers p, q and r such that

$10^{s-1} < p < 2.10^{s-1} < q < 4.10^{s-1} < r < 8.\ 10^{s-1}$;

it is clear that each of the numbers p, q, r has s digits.

For s = 1, we have four primes of one digit 2,3,5 and 7. The number of two digits primes is twenty–one, of three digit primes is 143. Thus there exist at least three prime numbers of a hundred digits each. R.M. Robinson has found three prime numbers of a hundred digits.

$81.2^{324} + 1,\ 63.2^{326} + 1,\ 35.2^{327} + 1.$

We do not know so far any prime number having a thousand digits, although we known there exist atleast three such numbers. Such prime numbers are called as Titanic primes.

1.4. TWIN PRIMES :

There arises a series of questions about the infinite sequence of consecutive prime numbers, i.e. the sequence 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, . . . only some of there question can be answered easily.

The smallest two prime numbers are 2 and 3. These are successive natural numbers. the question arises: if there are other successive natural number which are both prime; the answer is no. Because of two successive natural numbers one is even, and if it is >2 then it is composite.

There are many pairs of successive odd numbers which are both primes. for example 3 and 5, 5 and 7, 11 and 13, 17 and 19, 29 and 31 , 41 and 43. We call such pairs twin primes. There are 152, 892 such pairs of numbers less than 30 million.

The following BASIC program computes and prints all twin primes less than 1000

```
100        REM TWIN PRIMES LESS THAN 1000
120        DIM A [1000], B [400]
120        FOR  X = 2 TO 1000
130        LET A [X] = 0
140        NEXT X
150        LET C = 0
160        LET S = SQR (1000)
170        FOR B = 2 TO 1000
180        IF A [B] < 0 THEN 250
190        LET C = C + 1
200        LET B[C] = B
210        IF B>S THEN 250
220        FOR X = B TO 1000 STEP B
230        LET A [X] = –1
240        NEXT X
250        NEXT B
260        PRINT "TWIN PRIMES"
270        PRINT
280        FOR X = 2 TO C
290        IF B[X] –B[X–1] <> 2 THEN 310
300        PRINT B[X–1]; B [X]
310        NEXT X
320        END
```

RUN

```
            TWIN PRIMES
  3                          5
  5                          7
 11                         13
 17                         19
 29                         31
 41                         43
 59                         61
 71                         73
```

101	103
107	109
137	139
149	151
179	181
191	193
197	199
227	239
269	271
281	283
311	313
347	349
419	421
431	433
461	463
521	523
569	571
599	601
617	619
641	643
659	661
809	811
821	823
827	829
857	859

Long ago the question was asked whether the number of twin primes is infinite. We do not know the answer to this question.

Twin prime conjecture : There are infinitely many prime pair p, p+2. The argument which makes it plausible that

$$(6) \qquad P_2(x) \sim \frac{2c_2x}{(\log x)^2}$$

where $P_2(x)$ is the number of these pairs with $p \le x$ and

$$c_2 = \prod_{p>2}\left\{\frac{p(p-2)}{(p-1)^2}\right\} - \prod_{p\geq 3}\left\{1-\frac{1}{(p-1)^2}\right\}.$$

It is an unanswered question whether there are infinitely many pair of twin primes that is, pairs of successive odd integers p and p +2 which are both primes. Electronic computers have discovered 152, 892 pairs of twin primes less than 30,000,000 and twenty pairs between 10^{12} and 10^{12} +10,000. The largest to date twin primes are $107570463.10^{2250} \pm 1$ each 2259 decimal digits long were discovered in 1985.

The largest known pairs of twin primes are $1706595 \times 2^{11235} \pm 1$ and $571305 \times 2^{7701} \pm 1$ found in 1990 by B. Parady, J. Smith and S. Zarantonello. In 1998, Ray Ballinger found the largest twin primes $835335 \times 2^{39014} \pm 1$ having 11751 digits.

For every $x > 1$ let $\pi_2(x)$ denote the number of primes p such that p+2 is also prime and $p+2 \leq x$.

Brun announced in 1919 that there exists an effectively computable integer x_0 such that, if $x \geq x_0$, then $\pi_2(x) < 100x/(\log x)^2$. Further $\sum(1/p + 1/(p+2))$ converges which express scarcity of twin prime, even if there are infinitely many of them.

$B = (1/3 + 1/8) + (1/5 + 1/7) + \ldots . + (1/p + 1/p+2) + \ldots ..$

is called Brun's constant and the value of B is 1.90216054.

Growth of π_2 (x)

x	π_2 (x)
10^3	35
10^4	205
10^5	1224
10^6	8169
10^7	58980
10^8	440312
10^9	3424506
10^{10}	27412679

Record : The largest exact value for the number of twin primes below a given limit has been published by Brent in 1976.

$$\pi_2(10^{11}) = 224,376,048.$$

Now the question is: which digits can there be at the beginning and at the end of a prime number?

The last digit of a prime number having more than one digit can not be even because then the number would be >2 and thus even and composite; the last digit can not be 5 because then the number would be greater than 5 and divisible by 5 and so it would be composite. Thus the last digit of a prime number >10 can only be 1, 3, 7 and 9.

There exists prime number having at the beginning and at the end an arbitrary large number of digits equal to 1 (but the middle digit may be other than 1).

In this connection the problem arises whether there exist infinitely prime numbers whose digits are all 1, for example11 and

11,111,111,111,111,111,111,111, = $(10^{23} - 1) / 9$.

Such type of numbers are caled as Repunits. Repunits having 9 ones is prime.

The proof that the last number is prime is not easy. However it is easy to prove that, if a number whose digits are all 1 is prime then the number of its digits must be prime. This property, however, is not sufficient because, for example,

111= 3 x 37; 11,111=41 x 271; 1, 111, 111 = 239 x 4649.

Also the number $(10^{37}-1) / 9$ having thirty seven digits is composite and the number $(10^{641} - 1) / 9$ having 641 digits is composite and divisible by 1238.

Prime numbers other than those formed by the same digits have been found which remain prime after every permutation of their digits, for example 13,113. It is not known whether these numbers are finite.

Also we do not know if there exist infinitely many prime numbers whose first and last digits are 1 and the remaining ones are 0, as for example the number 101. It is easy to prove that such a prime number must be of the form $10^{2^n} + 1$ where n is natural number, but again this property is not sufficient because $10^{2^2} + 1 = 73 \times 137$ we can not answer the question whether the sequence of sums of the digits of consecutive prime number tend to infinity.

1.5. CONJECTURE OF GOLDBACH

In 1742, Christian Goldbach stated conjecture that each even number >2 is the sum of two primes This conjecture still remains neither proved nor disproved. A stronger conjecture has been made, namely that every even number > 6 is the sum of two distinct prime numbers and this is verified for numbers < 100,000.

It can be proved that the last conjecture is equivalent to the statement that every natural number > 17 is the sum of three different primes for example 19 = 3+5+11. A. Schinzel has proved that the conjecture of Goldbach implies that every odd number > 17 is the sum of three different primes . Every even integer greater than 4 can be written as a sum of two odd prime numbers.

Example : $10 = 3 + 7 = 5 + 5$

$22 = 3 + 19 = 5 + 17 = 11 + 11$

$30 = 7 + 23 = 11 + 19 = 13 + 17.$

THEOREM 16 : Every odd number > 7 is the sum of three odd primes.

PROOF : From the conjecture of Goldbach this follows easily. If n is a natural number and $2n + 1 > 7$ then $2n + 1 - 3 = 2(n-1) > 4$. The even number $2(n-1) > 4$ is, by the conjecture of Goldbach, the sum of two prime p and q, which can not be even, because our number is > 4. The prime numbers p and q are therefore odd and the number $2n + 1 = 3 + p + q$ is the sum of three odd primes.

We do not know whether every odd number >7 is the sum of three odd primes, but in 1937, I. Vinogradov proved that every sufficiently large odd number is the sum of three odd primes. We know a number $a = 3^{3^{15}}$ which is the sum of three odd primes.

THEOREM 17 (Euler) :

The following two statements are equivalent.

(*A*) Every integer n > 5 is the sum of three primes.

(*B*) Every even integers $2n \geq 4$ is the sum of two primes.

PROOF : Assume that (*B*) holds and if $2n \geq 6$, then $2n - 2 = p + p'$ so $2n = 2 + p + p'$, where p, p' are primes. Also $2n + 1 = 3 + p + p'$ which proves (*A*).

Conversely, if (*A*) is assumed to be true and if $2n \geq 4$, then $2n + 2 = p + p' + p''$ with p, p', p'' primes, then necessarily $p'' = 2$ (say) and $2n = p + p'$.

Note that it is trivial that (*B*) is true for infinitely many even integers

$$2p = p + p \text{ (for every prime).}$$

Further it has been proved that every natural number >11 is the sum of two or more different primes. For example $12 = 5 + 7$, $13 = 2 + 11$, $17 = 2 + 3 + 5 + 7$, $29 = 3 + 7 + 19$. A Makowski has proved that every natural number > 55 is the sum of different primes of the form $4k + 3$.

THEOREM 18 : Every odd (positive or negative) integer can be written in an infinite number of ways in the form $p + q - r$ where p, q and r are odd primes.

PROOF : For every integer k there exists an odd prime number r such that $2k - 1 + r > 4$. (It is enough to take for r a sufficiently large prime number). But then $2k - 1 + r$ is an even number > 4, therefore, by Goldbach's conjecture

$2k - 1 + r = p + q$, where p and q are odd prime numbers. Hence $2k - 1 = p + q - r$ in which the prime number r may be arbitrarily large. Hence proved

THEOREM 19 : Every natural number >11 is the sum of two composite numbers.

PROOF : If $n > 11$ is an even number, then $n - 4$ is an even number >2, i.e. the sum of two composite numbers. But if $n > 11$ is odd then $n - 9$ is an even number >2, i.e. it is composite and $n = (n - 9) + 9$, i.e. the sum of two composite number

Remark : From the above theorem we should not conclude that the inquiry into composite numbers is easier than investigations about prime numbers, for it is still unanswerable whether or not among numbers $F_n = 2^{2^n} + 1$, n = 1, 2, 3... we have infinitely many composite numbers. We only know thirty eight such composite number of which the greater is F_{1945}.

G.H. Hardy and J. E. Littlewood conjectured that every sufficiently large natural number which is not a perfect square is the sum of a square of an integer and a prime number.

THEOREM 20 : There exist infinitely many squares of natural numbers which are , as also those which are not the sum of a prime number and the square of an integer.

PROOF : If p is an odd prime, then $(p + 1) / 2$ is a natural number and we have

$$\left(\frac{p+1}{2}\right)^2 = \left(\frac{p-1}{2}\right)^2 + p$$

On the other hand if $n = 3k + 2$, where k is a natural number, then for some integer x and a prime number p we can not have,

$$n^2 = x^2 + p,$$

for then n would be .x and

$$p = n^2 - x^2 = (n + x)(n - x)$$

whence, considering that p is prime, $n - x = 1$ and $n + x = p$, so that

$$p = 2n - 1 = 3(2k + 1)$$

which is not possible for a natural number k.

1.6. FERMAT NUMBERS.

Fermat numbers are numbers of the form $F_n = 2^{2^n} + 1$, where $n = 0, 1, 2$

A famous mathematician of the seventeenth century, P. Fermat conjectured that all these numbers are prime. This is true for n = 0, 1, 2, 3, 4. But L. Euler in 1732 showed that the number,

$$F_5 = 2^{2^5} + 1 = 4,294,967,297$$

having 10 digits is composite and divisible by 641. We know 38 composite numbers Fn, namely for n=5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 18, 19, 23, 36, 38, 39, 55, 58, 63, 73, 77, 81, 117, 125, 144, 150, 207, 226, 228, 260, 267, 268, 284, 316, 452, 1945.

These 38 composite numbers F_n include those for which we know the prime factors (for example F_5 and F_6) those whose prime factor we do not know. The number of digits of the composite Fermat number F_{1945} is $>10^{582}$

(1) The largest know Fermat prime is F_4=65537

(2) The largest known composite Fermat number is F_{23471} (Keller 1984, Amonwin 1985) which has a factor $5 \times 2^{23473} + 1$ and more than 10^{7000} digits. Keller has also shown that F_{9448} is composite having the factor $19 \times 2^{9450} + 1$.

$F_{10} = 455925777 \times 6487031809 \times C_{291}$ where C_m denotes a composite number with m digits.

To find the prime divisor of Fn we know the following: Every natural divisor of the number F_n must be of the form $2^{2n+2}k+1$, where k is an integer ≥ 0. This can be verified by the following examples.

Example 1 : If n = 1945, the divisors of F_{1945} can only be in the arithmetic progression $2^{1947} k + 1$ (k = 0, 1, 2..). For k = 0 we obtain the trivial divisor 1. For k = 1, the number $2^{n+2} + 1 = 2^{1947} + 1$ is obviously divisible by 3 and so is not a prime. For k = 2, the number $2^{n+2} . 2 + 1 = 2^{1948} + 1 = (2^4)^{487} + 1$ is divisible by $2^4 + 1$ and so is not prime. For k = 3, the number $2^{n+2}. 3 + 1 = 2^{1947}.3 + 1$ is composite, divisible by 5 because $5 \mid 2^4 - 1$ when $5 \mid 2^{1944} - 1$ and if we multiply the right side by $2^3.3$, $5.2^{1947}.3 - 24$, whence $5 \mid 2^{1947}.3 + 1$. For k = 4, the number $2^{n+2}. 4 + 1 = 2^{1949} + 1$ is divisible by 3 and so is composite.

Thus trying to find the prime divisor of the number F_{1945}, we must divide it by $2^{1947}.5 + 1 = m$. As the division turns out to be without remainder, it follows that m is the smallest divisor of F_{1945} is a prime.

Example 2 : The divisor of F_5 is $2^7 . k + 1$ that is 128 k + 1. For k = 1, we get 129 which is composite. For k = 2 we get 257 which is a prime but does not divide F_5 = 4,294,467,297. For k = 3 we get 385 which is composite. For k = 4 we get $513 = 2^9 + 1$ is divisible by 3 hence composite. For k = 5 we get the prime number 641 which divides F_5. Dividing F_5 by 641 we get the quotient

6,700,417 which is also a prime. If F_5 is composite it must have a prime divisor not greater than its square root and so less than 2600. So we have $128\,k + 1 < 2600$ whence $k < 21$. For F_6 the prime divisor is $2^8 \,.\, 1071 + 1$. Thus F_6 is composite.

THEOREM 21 : If $a \geq 2$ and $a^n + l$ is prime, then a is even and $n = 2^m$.

PROOF: For if a is odd then $a^n + l$ is even and if n has an odd factor k and $n = kl$, then $a^n + l$ is divisible by $a^l + l$ and

$$\frac{a^{kl}+1}{a^{l}+1} = a^{(k-1)l} - a^{(k-2)l} + \ldots\ldots + 1$$

Hence $n = 2^m$. Hence proved.

THEOREM 22 : If F_n is prime, the number $3^{2^{2^n\;1}} + 1$ is divisible by F_n.

For the proof of this theorem we require the following lemma.

LEMMA : If k is a nonnegative integer and the number $p = 12k + 5$ is prime, then the number $3^{6\,k+2} + 1$ is divisible by p.

PROOF : The lemma is true for the number $k = 0$; we may therefore say that k is a natural number. Let $p = 12k + 5$. We consider the product of the first $6k + 2$ natural numbers divisible by 3 and divide the factors of the product of three groups, putting in the first group the first 2k factors, in the second next $2k + 1$ factors and in the third the remaining $2k + 1$ factors.

First group gives the product 3. 6. 9…6k.

The second group gives the product $(12k + 3)\, 12k\, (12k-3) \ldots (6k + 6)\, (6k + 3)$

(Changing the order of factors) which , since $p = 12k + 5$, can be written in the form

$$(p-2)\,(p-5)\,(p-8)\,\ldots.\,[\,p-(6k + 2)\,].$$

Because the number of divisors is odd i.e. $2k + 1$, our product, after expanding and collecting the terms divisible by p, gives us the number $p\,u - 2.5.8.\,(6k + 2)$, where u is a certain integer.

The factors of third group give the product

$$(12k + 6)\,(12k+9)\,(12k + 12)\,\ldots\,(18k + 6)$$
$$= (p+1)\,(p+4)\,(p+7)\,\ldots\,(p+6k + 1)$$
$$= p\,v + 1.4.7.\,.\,.\,.\,.\,(6k + 1)$$

where v is a natural number.

We have now

$$3.6.9\ldots(18k+6)$$
$$= 3.6.9\ldots. 6k\,(p\,u - 2.5.8\ldots(6k+2)\,(p\,v + 1.4.7.(6k+1))$$
$$= p\,w - 1.2.3.4.5.6\ldots.(6k+1)\,(6k+2)$$
$$= p\,w - (6k+2)\,!$$

where w is an integer.

But $3.6.9\ldots(18k+6) = (6k+2)!\,3^{6k+2}$

i.e. $(6k+2)!\,3^{6k+2} = p\,w - (6k+2)!$

i.e. the number p w is divisible by (6k+2)!, and so

$$p\,w = (6k+2)!\,t,$$

where t is an integer. But $6k+2 < 12k+5 = p$, and so the number (6k+2)! is not divisible by p. As the product (6k+2)! t is divisible by p, t must be divisible by p, t = p s, where w= (6k+2)! s, where s is an integer. Hence $3^{6k+2} = ps - 1$, when it follows that the number $3^{6k+2} + 1$ is divisible by p. Hence proved.

PROOF OF THE THEOREM 22 : Let n be a given natural number. We have $2^n = 2m$ where m is a natural number. Hence $F_n - 1 = 4^m$ from which it follows that the number $F_n - 5$ is divisible by 4. We have $F_n - 1 = 4^m = (3+1)^m = 3t + 1$, where t is a natural number. Hence $F_n - 5 = 3(t-1)$ which proves that $F_n - 5$ is divisible by 3, and since as we have shown it is divisible by 4, it is divisible by 12. So $F_n = 12k + 5$, where k is an integer. From the lemma it follows that if F_n is prime then $3^{6k+2} + 1 = 3^{(Fn-1)/2} + 1 = 3^{2^{2^n - 1}} + 1$ is divisible by F_n. Hence proved.

Example : F_7 is composite.

To prove this it is enough to show that $3^{2^{127}} + 1$ is not divisible by

F_7 = 340, 282, 366, 920, 938, 463, 374, 607, 431, 768, 211, 457 . To find this we have to find the remainder of the division of $3^{2^{127}}$ by F_7 .The number 3^{2^7} has 61 digits and calculate the remainder r of its divisor by F_7. The remainder r_1 of the divisor r^2 by F_7 will be the division of 3^{2^8} by F_7. Similarly the remainder r_2 of the division of $r_1{}^2$ by F_7 will be the remainder of division of 3^{2^8} by F_7. Proceeding we arrive at the remainder r_{120} of the division of the $3^{2^{127}}$ by F_7. In this way it is found that $r_{120} \neq 2^{2^7}$, whence it follows that the number $3^{2^{127}}+1$ is not divisible by F_7. So by the theorem 22, F_7 is composite. For the number F_{16}, the smaller prime divisor is 2^{18}. 3150 +1 was found in 1953, and conjectured that all number of the sequence .

$$2+1,\ 2^2+1, 2^{2^2}+1,\ 2^{2^{2^2}}+1, \ldots.$$

are prime was disproved. We do not know, if in this sequence there are infinitely many prime numbers or infinitely many composite numbers.

1.7. PRIME NUMBERS OF THE FORM $n^n+1, n^{n^n}+1$ etc.

Like Fermat numbers discussed in ξ 1.6. the question arises how many prime numbers are there of the form $n^n + 1$, where n is a natural number and it must be

$$\geq F_{20} > 2^{2^{20}} > 2^{10^6} > 10^{3.10^5}$$

So among the numbers $n^n + 1$, there are only three primes:

$1^1 + 1 = 2$, $2^2 + 1 = 5$, $4^4 + 1 = 257$.

Consider the number of the form $n^{n^n}+1$.

We have $1^{1^1} = 2$, $2^{2^2} + 1 = 17$. As above we can prove that if the number $n^{n^n}+1$, where n is a natural number >1, is a prime, then for same integer $r \geq 0$, we must have $n = 2^{2^r}$, so that

$$n^{n^n} + 1 = F_{r+2}r + 2^{r}$$

For r = 0, since $F_2 = 17$, for r = 1 the number F_9 is composite . For r = 2, F_{66} is composite. Hence we have

Among the numbers having not more then 10^{18} digits, there exist only two prime numbers of the form $n^{n^n}+1$, where n is a natural number 2 and 17.

Now the question arises:

Among the numbers,

(*A*) $2 + 3, 2^2 + 3, 2^{2^2} + 3, \ldots..$ how many primes exist.

The answer is that there are only a finite number of primes in the set (*A*).

Suppose that $n^n + 1$ is prime for natural number n. Each natural number n is of the form $n = 2^k \cdot m$, where k is an integer ≥ 0, and m is an odd number.

If m >1, then the number $n^n + 1 = \left(n^{2^k}\right)^m + 1$ would be $> n^{2^k} + 1$ and is divisible by $n^{2^k} + 1$, giving $n^n + 1$ to be composite. Therefore, m must be equal to 1. So $n = 2^k$.

If $k = 0$, then $n - 1$, and the number $n^n + 1$ is a prime. If $k > 0$, then $k = 2^r.s$, r is an integer ≥0 and s is an odd number.

If s >1, then $n^n + 1 = 2^{2^r s.n} + =$ is divisible by $\left(2^{2^r.n}\right) + 1$, hence composite. Therefore, $s = 1$ and $k = 2^r$ and $n = 2^r$ and

$$n^n + 1 = 2^{2^r.2^{2^r}} + 1 = 2^{2^{r+2^r}} + 1 = F_{r+2^r}.$$

Hence, $n^n + 1$ is prime if F_{r+2} r is prime.

For $r = 0$, since $F_1 = 5$ is prime we get $2^2 + 1 = 5$ is prime. For $r = 1$, since $F_3 = 257$ is prime we get the prime number $4^4 + 1 = 257$. For $r = 2$, F_6 is composite divisible by $2^8.1071 + 1$. We do not get n so that $n^n + 1$ is prime. For $r = 3$ since F_{11} is composite $n^n + 1$ is also composite. If, therefore besides the number 2, 5 and 257 there exist prime numbers of the form $n^n + 1$,

then they must be $\geq F_{20} > 2^{2^{20}} > 2^{10^6} > 10^{3.10^5}$.

THEOREM 23 : Among the numbers $2^{2^2} + 5, 2^{2^{2^2}} + 5, \ldots..$ there are no primes, because each of these numbers is divisible by 7.

PROOF : For a natural number k, the number $2^{2k} = (3+1)^k$ where divided by 3 gives the remainder 1, so $2^{2k} = 3t + 1$, where t is a natural number.

Hence $2^{2^{2k}} + 5 = 2^{3\,t+1} + 5 = (7+1)^t\, 2 + 5$ which is divisible by 7.

THEOREM 24 : For Fermat numbers F_n and F_m, where $m > n \geq 0$, gcd $(F_m, F_n) = 1$.

PROOF : Put $d = \gcd(F_m, F_n)$. Since Fermat numbers are odd integers, d must be odd . If

we set $x = 2^{2^n}$ and $k = 2^{m-n}$, then

$$\frac{Fm - 2}{Fn} = \frac{(2^{2^n})^{2^{m-n}} - 1}{2^{2^n} + 1} = \frac{x^k - 1}{x + 1} = x^{k-1} - x^{k-2} + \ldots\ldots\ldots$$

where $F_n \mid (F_m-2)$. From $d \mid F_n$, it follows that $d \mid (F_m-2)$. But $d \mid F_m$ implies $d \mid 2$.

But d is an odd integer and so d = 1. This proves the theorem .

THEOREM 25 : For $F_n = 2^{2^n} +1$ we have $F_n | 2^{F_n} -2$ $(n = 1, 2, 3, \ldots)$
PROOF : By induction we can show that $2^n \geq n+1$ which implies that

$$2^{n+1} \Big| 2^{2^n} \text{ and } 2^{2^{n+1}} - 1 \Big| 2^{2^{2^n}} - 1$$

Therefore

$$F_n = 2^{2^n} + 1 \Big| 2^{2^{n+1}} - 1 \Big| 2^{2^{2^n}} - 1 | 2^{2^{2^n}+2} - 2$$

$$= 2^{F_n} - 2 \text{ and } F_n \mid 2^{F_n} - 2$$

Hence proved.

1.8. MERSENNE NUMBERS

Mersenne numbers are numbers of the form $M_n = 2^n - 1$, where n = 1, 2, 3,.. These numbers are interesting in two respects. Firstly the greatest known prime numbers are Mersenne numbers and secondly it gives rise to perfect number, that is those which are equal to the sum of their natural divisors less than those numbers themselves.

Since $1+2+2^2 + \ldots . 2^{n-1} = 2^n - 1$. i.e. the Mersenne number is the sum of the first n terms of the G.P.

The following is the BASIC program for finding Mersenne number

```
100            REM MERSENNE PRIMES
110            PRINT "PRIME" "MERSENNE"
120            PRINT "NUMBER", "PRIMES"
130            PRINT
140            FOR K = 1 TO 8
150            READ P
160            LET M = 2↑P–1
170            PRINT P,M
180            NEXT K
190            DATA 2,3,5,7,13,17,19
200            END
```

RUN

PRIME NUMBER	MERSENNE PRIMES
2	3
3	7
5	31
7	127
13	8191
17	131071
19	524287

THEOREM 26 : If n is composite then the number in M_n is composite

PROOF : If n = a b where a and b are natural numbers > 1, then $2^a - 1 > 1$ and $2^n - 1 = 2^{ab} - 1 > 2^a - 1$, and the number $2^{ab} - 1$ is divisible by $2^a - 1$, and so is composite. Therefore if the number M_n, where $n > 1$, is prime, then the number n must be prime, but the converse is not necessarily true because, for example

$$M_{11} = 2^{11} - 1 = 2047 = 23 \times 89.$$

If p is a prime number, then each natural divisor of the number M_p must be of the form $2kp + 1$, where k is an integer ≥ 0.

Many of the Mersenne numbers M_p, where p is a prime number, are composite. For example $47 \mid M_{23}$, $167 \mid M_{83}$ $263 \mid M_{131}$, $359 \mid M_{179}$.

Conjecture 1. Among the numbers M_p, where p is a prime number, there exists infinitely many which are composite.

M_n, for n =2, 3, 5, 7, 13, 17, 19, 31, 61, 89, 107, 127, 521, 607, 1279, 2203, 2281, 3217, 4253 and 4423 is prime.

Conjecture 2. If the Mersenne number M_n is a prime then the number M_{M_n} is also prime.

This is true for first four smallest Mersenne numbers, but for fifth Mersenne number $M_{13} = 8191$ it is not true because $M_{M_{13}} = 2^{8191} - 1$ is composite (Robinsen). The verification of this required one hundred hours work on an electronic computer. Again although M_{17}, M_{19} are prime $M_{M_{17}}$ and $M_{M_{19}}$ are composite.

Let σ (N) denotes the number of positive divisors of N. If σ (N) = 2N then N is said to be a perfect number. Example N = 6, N = 28 are perfect numbers.

The following BASIC program computes the first two perfect numbers.

```
10          FOR N = 2 TO 100
20          LET S = 0
30          FOR D = 1 TO N/2
40          IF INT (N/D) <> N/D THEN 60
50          LET S = S+D
60          NEXT D
70          IF S <> N THEN 90
80          PRINT N; "IS A PERFECT NUMBER"
90          NEXT N
99          END
```

```
                RUN
 6  IS A PERFECT NUMBER
28 IS A PERFECT NUMBER.
```

THEOREM 27 : All even perfect numbers are of the form $2^{p-1} M_p$, i.e. $2^{p-1}(2^p - 1)$ if $M_p = 2^p - 1$ is a prime number.

PROOF : Let N be a number such that $N = 2^{p-1} M_p = 2^{p-1}(2^b - 1)$.

To prove that N is a perfect number.:

If $\sigma(N)$ denotes the sum of the divisors of N, then

$$\begin{aligned}\sigma(N) &= \sigma(2^{p-1}(2^p - 1) = \sigma(2^{p-1})\,\sigma(2^p - 1)\\ &= (2^p - 1)(2^p - 1 + 1)\\ &= 2(2^{p-1}(2^p - 1))\\ &= 2N.\end{aligned}$$

making N a perfect number.

Conversely assume that N is an even perfect number. We may write N as

$N = 2^{p-1} m$, where m is an odd integer. Then

$$(7)\quad \sigma(N) = \sigma(2^{p-1} m) = \sigma(2^{p-1})\,\sigma(m) = (2^p - 1)\,\sigma(m)$$

Since N is perfect

$$(8)\quad \sigma(N) = 2N = 2^p m.$$

Equating (7) and (8) $2^p m = (2^p - 1)\,\sigma(m)$

which is simply to say that $2^p - 1 \mid 2^p m$.

But 2^p-1 and 2^p are :elatively primes, whence $2^p-1 | m$, say $m=(2^p-1) M$ Substituting in (8) we get.

$$2^p(2^p-1) M = (2^p-1)\sigma(m)$$

This implies $\sigma(m) = 2^p M$. Since m and M are both divisors of m (with $M < m$) we have $2^p M = \sigma(m) \geq m + M = 2^p M$

leading to $\sigma(m) = m + M$. The implication of this inequality is that m has only two divisors M and m itself. It must be that m is prime and M = 1, in other words $m = (2^p-1) M = 2^p-1$ is a prime number.

THEOREM 28 : If $a^k - 1$ is prime ($a > 0, k \geq 2$) then a = 2 and k is prime p.

PROOF : Now $a^k-1 = (a-1)(a^{k-1} + a^{k-2} + \ldots\ldots + a+1)$

$a^{k-1} + a^{k-2} + \ldots + a+1 \geq a+1 > 1$.

Since a^k-1 is prime, $a-1 = 1$ so that $a = 2$.

If k were composite, then take $k = r\,s$, with $r > 1$ and $s > 1$.

Thus $a^k-1 = (a^r)^s - 1 = (a^r - 1)(a^{r(s-1)} + a^{r(s-2)} + \ldots + a^r + 1)$

and each factor on the right is greater than 1.

But this violates the primality of a^k-1, so k must be prime.

For p = 2, 3, 5, 7 the values 3, 7, 31, 127 of 2^p-1 are primes so that

$$2(2^2-1) = 6$$
$$2^2(2^3-1) = 28$$
$$2^4(2^5-1) = 496$$
$$2^6(2^7-1) = 8128.$$

All are perfect numbers, are even and end in the digit 6 or 8. No odd perfect number exists. As only 38 Mersenne primes are known, only 38 even perfect numbers are known till to day. The 31st Mersenne prime is M_{216091}. The largest known even perfect number, the 31st one is

$$P_{31} = 2^{216090}(2^{216091} - 1).$$

a number of 130,100 digits.

$P_{32} = 2^{756838}(2^{756839}-1)$ has 455663 digits. Most recently the 38th Mersenne prime was discovered by Nayan Hajratwala in June 1999. The number is $2^{6,972,593} - 1$. Just as for Fermat numbers, there are many open problems about Mersenne numbers

(1) Are there infinitely many Mersenne primes?

(2) Are there infinitely many composite Mersenne numbers?

Fermat number is $F_n = 2^{2^n} + 1$ ($n \geq 0$) and a triangular number is $T_m = ½\, m\,(m+1)$.

By a triangular number we mean the numbers, which are represented by vertices of a triangle. For example 1, 3, 6, 10, 15, 21, 28 are triangular numbers.

Exercise : Show that the only Fermat number which is also triangular number is 3.

Solution : If F_n is triangular then

$$2^{2^n} + 1 = ½\, m\,(m+1)$$

$$\Rightarrow 2^{2^n} = ½\, m\,(m+1) - 1$$

$$\Rightarrow \quad 2^{2^n} + 1 = m\,(m+1) - 2 = m^2 + m - 2 = (m-1)\,(m+2).$$

Both (m + 2) and m −1 are power of 2 and the difference is 3, so m=2 is the only solution i.e. 3 is the only triangular number which is also a Fermat number.

A number a is a triangular number it is necessary that 1+ 8a should be a perfect square .

THEOREM 29 : No Fermat number other than 3 is a triangular number.

PROOF : $2^{2^n} + 1$ is a triangular number

$$\Rightarrow 1 + 8\,(2^{2^n} + 1), \text{ i.e. } 9 + 2^{2^n+3} \text{ is a perfect square.}$$

Let $9 + 2^{2^n+3} = M^2$ and $2^n + 3 = t$, then $2^t = M^2 - 9 = (M+3)\,(M-3)$

(*A*) $\Rightarrow M + 3 = 2^s$ and $M - 3 = 2^l$ and $s + l = t$, l can not be equal to 0. Since if l= 0, $2^s = 7$ which is absurd since s is an integer.

Hence $s > l \geq 1$.

Eliminating M from (*A*) we get

$$6 = 2^s - 2^l = 2^l\,(2^{s-l} - 1)$$

$$\text{i.e. } 2^l = 2 \text{ or } l = 1,\ 2^{s-l} - 1 = 3.$$

Hence s s = 3 and t = 4.

Since $2^n + 3 = 4 \Rightarrow n = 0$.

n can not have any value other than 0. Hence the result.

THEOREM 30 : All Mersenne numbers for which n is odd and greater than one can not be triangular number.

PROOF : Mersenne number is $2^n - 1$. In order that it will be a triangular number,

$1 + 8(2^n - 1)$ must be a perfect square.

That is $1 + 2^3(2^n - 1)$ i.e. $2^{n+3} - 7$ is a perfect square.

That is $2^{n+3} - 7 = M^2$.

If n is odd, n + 3 is even. Let n + 3 = 2 k , then

$$2^{2k} - M^2 = 7 \quad \Rightarrow (2^k + M)(2^k - M) = 7.$$
$$\Rightarrow 2^k - M = 1, \; 2^k + M = 7$$
$$\Rightarrow 2^k + 2^k - 1 = 7,$$

or $2^k = 4$ i.e. $k = 2$ and $n = 1$.

Ref: M. Satyanarayan, Math Student 1968.

1.9. SOLUTION OF EQUATION IN PRIME NUMBERS :

We know many simple equations about which we do not know whether they have many solutions in prime numbers.

For example: The equation x + y = z. The solution of this equation in prime number x, y, z is equivalent to the question, whether there exists infinitely many pairs of twin primes, for, if p, q and r are prime numbers such that p + q = r, obviously the prime number p and q can not both be odd (because then their sum would be an even number > 2 and so composite). Therefore one of the numbers p and q, say q is even and so equal to 2. The numbers p and r = p + 2 would then be a pair of twin primes. If the number p and r = p +2 are a pair of twin primes then the numbers x = p, y = 2, z = p + 2 are primes and give the solution of the equation x + y = z.

Consider the equation 2x ± 1 = y. We do not know they have infinitely many solutions. The solution of 2x + 1 = y, (x, y) = (2, 5), (3, 7), (5, 11), (11, 23) and for 2x – 1 are (2, 3), (3, 5), (7, 13), (19, 37). Equations of the type $x + y = z + t$ and $x^2 + y^2 = z^2 + t^2$ have infinitely many solutions in distinct primes x, y, z and t.

1.10. MAGIC SQUARES :

A magic square consists of a series of integer arranged in a square so that the sum of the number in any row or column is always the same. Magic squares have been known since ancient times. The construction of magic squares is a favorite topic in recreational mathematics.

In early school days Ramanujan's attention was devoted in constructing magic squares.

The following are the examples of magic squares 3 x 3 with sum (*i*) r = 15 and

(*ii*) r = 27 where r is the sum of rows and columns.

6	1	8
7	5	3
2	9	4

r = 15

15	1	11
5	9	13
7	17	3

r = 27

Construction of magic squares :

Consider two sets of natural numbers. $S_1 = \{ A, B, C. . . \}$, $S_2 = \{ P, Q, R . . . \}$ each with n elements. Take the n^2 number in the direct sum $S_1 + S_2$ and arrange them in an n x n square so that each letter appears exactly once in each row, column and diagonal. Then we get a magic square.

Entry (*i*) : Let m_1 and $m_2$denote the sum of the middle row and middle column respectvely of a 3 x 3 square aaray of numbers. Let c_1 and c_2 denote the sums of the main diagonal and second diagonal, respectively. Let S denotes the sum of all nine elements of the square. Then if x denotes the centre element of the square

$$x = 1/3\ (m_1 + m_2 + c_1 + c_2 - S).$$

It is clear that

$$m_1 + m_2 + c_1 + c_2 = S + 3x$$

as x is counted four times in the left side.

Entry (*ii*) : Suppose that the sum of each row and column is equal to r. Then if x denotes the centre element of the square then

$$x = 1/3\ (c_1 + c_2 - r)$$

since by entry (*i*)

$$x = 1/3\ (r + r + c_1 + c_2 - 3r) = 1/3\ (c_1 + c_2 - r).$$

Note that if the square is magic, then entry (*ii*) implies that x = r/3 i.e. r is a multiple of 3.

Result : In a3 x 3 magic square the elements in the middle row, middle column, and each diagonal are in arithmetic progression.

PROOF : In each case the second element or middle one is r/3. If a and b are the first and third elements, respectively then a + r/3 + b = r

$$\Rightarrow b - r/3 = r/3 - a$$

i.e. three numbers are in arithmatical progression.

Example 1 : r = 15 middle term = 15/3 = 5, a + b = 10.

The values of a and b are the pairs (1, 9), (2, 8), (3, 7), (4, 6).

6	1	8
7	5	3
2	9	4

r = 15

Example 2 : Construct magic square with (*i*) r = 36 and all elements even and (*ii*) r = 63 and all elements divisible by 3.

Solution: In the first case x = 12 and in the second case x = 21.

14	4	18
16	12	8
6	20	10

r = 36

24	9	30
27	21	15
12	33	18

r = 63

1.11. DISTRIBUTION OF PRIME NUMBERS :

We now introduce a function π (x) which is a measure of distribution of primes amongst the natural number.

Definition : For each real number x, π(x) denotes the number of primes that do not exceed x, i.e.

$$\pi(x) = \sum_{p \leq x} 1.$$

Example : $\pi(-1) = \pi(1) = 0, \pi(2) = \pi(5/2) = 1.$

THEOREM 31 (PRIME NUMBER THEOREM) :

The prime number theorem asserts that

$$\lim_{x\to\infty} \frac{\pi(x)}{(x / \log x)} = 1.$$

In Chapter 4 we will do elementary proof of this prime number theorem.

1.12. SOME MORE NUMBERS AND THEIR COMPUTER PROGRAMMING

AMICABLE NUMBERS OR FRIENDLY NUMBERS : Let m and n be two integers such that $\sigma(m) - m = n$ and $\sigma(n) - n = m$, then m and n are said to be amicable pair. Here $\sigma(m)$ denotes the sum of proper divisor of m.

Example : The smallest amicable number s are 220 and 284 because

$$\sigma\ (220) = 1+2+4+5+10+11+20+44+55+110 = 284$$

and $$\sigma\ (284) = 1+2+4+71+142 = 220.$$

THEOREM 32 : A pair of integers (m, n) is amicable if and only if

(9) $$\sigma(m) = \sigma(n) = m + n.$$

PROOF : (*i*) Let $\sigma(m) = \sigma(n) = m + n$

Then $\sigma(m) - m = n$ and $\sigma(n) - n = m$.

Therefore (m, n) is an amicable pair.

(*ii*) Let (m, n) be an amicable pair then $\sigma(m) - m = n$ and $\sigma(n) - n = m$ Hence

$\sigma(m) = m + n$ and $\sigma(n) = n + m$. Hence proved.

There are about 400 amicable pairs of numbers known, of which some are.

220 and	284 (the smallest pair)
1184	1210
2620	2924
5020	5564
6232	6368
10744	10856
12285	14595

17296	18416
63020	76084
66928	66992
67095	71145
69615	87633
79750	88730
9 363 584	9 437 056
111 448 537 712	118 853 793 424

Several methods are available for finding amicable pairs. One common method is to let

$$A = (3)\,(2^{x}) - 1$$
$$B = (3)\,(2^{x-1}) - 1$$
$$C = (9)\,(2^{2x-1}) - 1.$$

If x is greater than 1, and A, B, and C are all primes, then 2^x AB and 2^xC constitute an amicable pair of numbers. For example, if x = 4, then A = 47, B =23, and C = 1151, which are all primes. Then

$$(2^4)\,(47)\,(23) = 17\,296$$

and

$$(2^4)\,(1151) = 18\,416.$$

The following is the BASIC program to produce the next pair of amicable numbers.

```
100    REM AMICABLE NUMBERS
110    FOR A = 1 TO 7000
120    LET S = 0
130    FOR D = 1`TO A/2
140    IF A/D <> INT (A/D) THEN 160
150    LET S = S+D
160    NEXT D
170    IF S <= A THEN 260
180    LET B = S
190    LET T = 0
200    FOR F = 1 TO B/2
```

```
210        IF B/F <> INT (B/F) THEN 230
220        LET T = T +F
230        NEXT F
240        IF T <.> a THEN 260
250        PRINT A; “AND”;B; “ARE AMICABLE NUM-
           BER”
260        NEXT A
270        END
```

RUN

```
220 and 284 are amicable numbers.
1184 and 1210 are amicable numbers.
```

1.13. ARMSTRONG NUMBERS

One hundred fifty three is an interesting number because

$$153 = 1^3 + 5^3 + 3^3.$$

Numbers such as this are called Armstrong numbers. Any N digit number is an Armstrong number if the sum of the N^{th} power of the digits is equal to the orginal number.

The following program finds three –digit Armstrong numbers.

```
100        REM ARMSTRONG NUMBERS
110        FOR N = 100 TO 999
120        LET A = INT (N/100)
130        LET B = INT (N/10) – 10*A
140        LET C = N –100*A–10*B
150        IF N <> A13 + B13 + C13 THEN 190
160        PRINT “ARMSTRONG NUMBER”;N
170        PRINT “EQUALS”; A↑3;” + “;B↑3;” + “;C↑3
180        PRINT
190        NEXT N
200        END
```

```
                RUN
ARMSTRONG NUMBER 153
EQUALS 1   + 125  + 27
ARMSTRONG NUMBER 370
 EQUAIS  27   +  343    +  0
ARMSTRONG NUMBER   371
EQUALS  64    +    0     +    343
```

1.14. LUCKY NUMBERS

A group of investigators working with Stanistav M. Ulam at Los Alamos scientific laboratories have discovered what they call the lucky number determined by a sieving process. As with the sieve of Eratosthenes, we begin by writing down all the natural numbers, in order, limiting ourselves to the first hundred to illustrate the process. If we leave 1 and strike out every second number, we eliminate all the even numbers.

1	3	~~5~~	7	9
~~11~~	13	15	~~17~~	~~19~~
21	~~23~~	25	~~27~~	~~29~~
31	33	~~35~~	37	~~39~~
~~41~~	43	~~45~~	~~47~~	49
51	~~53~~	~~55~~	~~57~~	~~59~~
~~61~~	63	~~65~~	67	69
~~71~~	73	75	~~77~~	79
~~81~~	~~83~~	~~85~~	87	~~89~~
~~91~~	93	95	~~97~~	99

In Eratosthenes sieve we next struck out every multiple of 3 because 3 was the next surviving number. The rule here is different: strike out every third number among those remaining. That means that 5 goes, and 11. 17. 23, etc. All such numbers are crossed out by a single slant line. The next surviving number is 7, so we let that stand and cross out every seventh remaining one (19,39,etc.) with two slant lines, to indicate what is happening . Then cross out every 9th , then every 13th, and so on. The slant lines indicates at what stage in the construction each number was eliminated.

1.15 ABUNDANT AND DEFICIENT NUMBERS

The number the sum of whose divisors is less than the number itself is called deficient, and a number exceeded by this sum is called abundant. As noted in section 1.8, the number is perfect when the sum of the divisors of a number, excluding the number itself, equals the number in question.

For example,

6 = 1 + 2 + 3 and is perfect.

12 < 1 + 2 + 3 + 4 + 6 and is abundant.

10 > 1 + 2 + 5 and is deficient.

The following BASIC profram factors a given number into its divisors determines whether the number is abundant, deficient, or perfect.

```
100    REM ABUNDANT AND DEFICIENT NUMBERS
110    PRINT "THIS PROGRAM WILL TAKE A NUMBER AND"
120    PRINT "COMPUTE THE SUM OF ITS DIVISORS"
130    PRINT
140    PRINT "TYPE THE NUMBER";
150    INPUT N
160    LET S = 0
170    PRINT "THE DIVISORS OF ";N;"ARE";
180    FOR X = 1 TO N–1
190    IF N/X <> INT 9N/X) THEN 220
200    LET S = S + X
210    PRINT X;
220    NEXT X
230    PRINT
240    IF S>N THEN 280
250    IF S<N THEN 300
260    PRINT N; "IS PERFECT"
270    GOTO 310
```

```
280          PRINT N; "IS ABUNDANT"
290          GOTO 310
300          PRINT N; "IS DEFICIENT"
310          PRINT
320          PRINT "TYPE 1 TO CONTINUE; 2 TO
             STOP";
330          INPUT Z
340          IF Z = 1 THEN 130
350          END
```

RUN

```
THIS PROGRAM WILL TAKE A NUMBER AND COMPUTE THE
SUM OF ITS DIVISORS

TYPE THE NUMBER ? 12

THE DIVISORS OF 12 ARE 1   2   3   4   6   12  IS ABUNDANT

TYPE 1 TO CONTINUE, 2 TO STOP ? 1
```

The following is the BASIC program for computing prime Number generators less than 400:

```
10          REM PRIME NUMBER GENERATOR
12          DIM A[400]
15          PRINT "PRIME NUMBERS"
20          LET R=1
25          LET A[1] =2
30          LET P=1
35          FOR X = 3 TO 400 STEP 2
40          FOR Y = 1 TO R
45          IF INT (X/A[Y] * A{Y] = X THEN 95
50          NEXT Y
55          LET R = R +1
60          LET S[R] = X
65          IF P>6 THEN 85
70          LET P = P+1
```

```
75          PRINT X;
80          GOTO 95
85          LET P=1
90          PRINT X
95          NEXT X
```

RUN

```
                PRIME NUMBER
  3     5     7    11    13    17    19
 23    29    31    37    41    43    47
 53    59    61    67    71    73    79
 83    89    97   101   103   107   109
113   127   131   137   139   149   151
157   163   167   173   179   181   191
193   197   199   211   223   227   229
233   239   241   251   257   263   269
271   277   281   283   293   307   311
313   317   331   337   347   349   353
359   367   373   379   383   389   397
```

The following is the BASIC program that will determines whether a given number positive integer is a prime number

```
100         REM IS THE NUMBER PRIME ?
110         PRINT "WHAT IS THE NUMBER";
120         INPUT N
130         IF INT (N) = N THEN 160
140         PRINT N; "IS NOT AN INTEGER"
150         GOTO 250
160         IF N > = 2 THEN 190
170         PRINT N; "IS LESS THAN 2"
180         GOTO 250
```

```
190        FOR 1 = 2 TO SQR (N)
200        IF INT (N/I) = N/I THEN 240
210        NEXT I
220        PRINT N; "IS A PRIME NUMBER"
230        GOTO 250
240        PRINT N; IS NOT A PRIME NUMBER"
250        PRINT
260        PRINT "TYPE 1 TO CONTINUE; 2 TO
           STOP";
270        INPUT C
280        IF C = 1 THEN 110
290        END
```

RUN

```
WHAT IS THE NUMBER ? 624
624 IS NOT A PRIME NUMBER

TYPE 1 TO CONTINUE; 2 TO STOP ?1
WHAT IS THE NUMBER / 769
769 IS A PRIME NUMBER
TYPE 1 TO CONTINUE; 2 TO STOP ? 1
WHAT IS THE NUMBER ? 1
1 IS LESS THAN 2

TYPE 1 TO CONTINUE ; 2 TO STOP ?1
WHATE IS THE NUMBER? 76,34
76, 34   IS NOT AN INTEGER

TYPE 1 TO CONTINUE; 2 TO STOP ? 1
WHAT IS THE NUMBER? 953
953 IS A PRIME NUMBER

TYPE 1 TO CONTINUE; 2 TO STOP ? 2
```

1.16. FIBONACCI NUMBERS

The sequence $u_1, u_2, u_3 \ldots. u_n$ in which $u_1 = 1, u_2 = 1, u_3 = 2$ and$(u_n = u_{n-1} + u_{n-2})$ for every $n \geq 2$ is called a Fibonacci sequence and it is termed as the Fibonacci number. The first few terms of the sequence are 1, 1, 2, 3, 5, 8, 13 …as

$$u_4 = u_3 + u_2 = 2+1 = 3$$

$$u_5 = u_4 + u_3 = 3 + 2 = 5$$

$$u_6 = u_5 + u_4 = 5+ 3= 8$$

and so on.

THEOREM 33 : For the Fibonacci sequence $\{u_n\}$, gcd $(u_n, u_{n+1}) = 1$ for every $n \geq 1$.

PROOF : Let us suppose that the integer $d > 1$ divides both u_n and u_{n+1}. Then their difference $u_{n+1} - u_n = u_{n-1}$ will also be divisible by d. From this and from the relation $u_n\ u_{n-1}\ u_{n-2}$ it may be concluded that $d|u_{n-2}$. The same argument shows that $d|u_{n-3}$, $d|u_{n-4}$, and that $d|u_1$. But $u_1 = 1$ This contradicts our assumption that $d > 1$. hence $d = 1$.

The question is whether u_n is prime for n prime. The answer is no, because

$$u_{19} = 4181 = 37.113.$$

THEOREM 34 : The greatest commen divisor of two Fibonacci numbers is again a Finonacci number, specially

$$\text{gcd}\ (u_m, u_n) = u_d, \text{ where } d = \text{gcd}\ (m, n).$$

PROOF : Assume that $m \geq n$ By Euclidean Algorithm we get

$$m = q_1\ n + r_1\ , \quad 0 < r_1 < n$$

$$n = q_2\ r_1 + r_2\ , \quad 0 < r_2 < r,$$

$$r_1 = q_3\ r_2 + r_3\ , \quad 0 < r_3 < r_2$$

.

.

$$r_{n-2} = q_n\ r_{n-1} + r_n, \quad 0 < r_n < r_{n-1}$$

$$r_{n-1} = q_{n+1}\ r_n + 0.$$

$$\begin{aligned} \text{gcd}\ (u_m, u_n) &= \text{gcd}\ (u_{q,n+r1}\ , u_n) \\ &= \text{gcd}\ (u_{q1n-1}\ u_r + u_{qn1}\ u_{r+1+,}\ , u_n) \\ &= \text{gcd}\ (\ u_{q1\ n-1}\ u_{r1}, u_n). \end{aligned}$$

Since gcd (a+c, b) = gcd (a,b),

$$\text{gcd}\ (u_m, u_n) = \text{gcd}\ (u_{r1}, u_n)$$

$= \gcd (u_{r1}, u_{r2})$

$= \ldots\ldots = \gcd (u_{rn-1}, u_{rn}),$

Since $r_n | r_{n-1}$, we have $u_{rn} | u_{rn-1}$,

whence $\gcd (u_{rn-1}, u_{rn}) = u_{rn}$.

But r_n, being the last non zero remainder in the Euclidean Alogorithm for m and n, is equal to gcd (m, n) we get

$\gcd (u_m, u_n) = u_{\gcd (m, n)}$.

Hence proved.

Corollary : In the Fibonacci sequence, $u_m | u_n$ if and only if $m|n$ for $m \geq 2$.

PROOF : If $u_m | u_n$, then $\gcd (u_m, u_n) = u_m$.

But $\gcd (u_m,u_n) = u_{\gcd (m,n)}$.

This implies that gcd (m, n) = m. From which it follows that m | n.

Here we give a BASIC program that computes and prints 30 Fibonacci numbers.

```
100        REM FIBONACCI NUMBERS
110        DIM F [30]
120        PRINT "FIBONACCI NUMBERS"
130        PRINT
140        LET F[1] = 1
150        LET F[2] = 1
160        FOR N = 1 TO 28
170        LET F[N+2] = F[N+1] + F[N]
180        NEXT N
190        REM PRINT 30 FIBONACCI NUMBERS
200        FOR X = 1 TO 30
210        PRINT F[X]
220        NEXT X
230        END
```

```
           RUN
------------------------
   FIBONACCI NUMBERS
             1
             1
             2
             3
             5
             8
            13
            21
            34
            55
            89
           144
           233
           377
           610
           987
          1597
          2584
          4181
          6765
         10946
         17711
         28657
         46368
         75025
        121393
        196418
        317811
        514229
        832040
```

1.17. FIBONACCI NUMBERS AND PRIMES

The BASIC program in the last section produced the first thirty Fibonacci numbers. As you have observed, all of these Fibonacci numbers are integer quantities, and some are prime numbers.

Let us now consider the problem of generating Fibonacci numbers and identifying those that are primes. An outline of the program procedure is :

1. Set F1 and F2 to 1 (F1 is the first Fibonacci number (F_{n-2}) and F2 second Fibonacci number (F_{n-1}).
2. Print F1 and F2 identifying each as a prime number.
3. Perform the following calculations for I = 3, 4,… 25.
   (*a*) Calculate a value for F using the formula F = F1 + F2.
   (*b*) Test to see if F is a prime number.
   (*c*) If F is a prime , identify it as such.
   (*d*) Update F1 and F2 in preparation for calculating a new Fibonacci number (assign the current value of F1 to F2, then assign the value of F to F1)

A BASIC program corresponding to the previous procedure follows :

```
100    REM FIBONACCI AND PRIME NUMBERS
110    PRINT “HOW MANY FIBONACCI NUMBERS”;
120    INPUT N
130    PRINT
140    PRINT
150    PRINT ‘FIBONACCI AND PRIME NUMBERS”
160    PRINT
170    LET F1=1
180    LET F2=1
190    PRINT “I=”; 1, “F=”;1,” (PRIME NUMBER)”
200    PRINT “I=”; 2, “F=”; 1,” (PRIME NUMBER)”
210    FOR I= 3 TO N
220    LET F= F1 +F2
230    FOR J=2 TO F–1
240    LET Q=F/J
250    LET Q1=INT (Q)
260    IF Q=Q1 THEN 300
270    NEXT J
```

```
280          PRINT "I="; I. "F="; F; "(PRIME NUMBER)"
290          GOTO 310
300          PRINT "I="; I, "F=";F
310          LET F2=F1
320          LET F1=F
330          NEXT I
340          END
```

RUN

```
HOW MANY FIBONACCI NUMBERS?24
FIBONACCI AND PRIME NUMBERS
     I = 1          F = 1          (PRIME NUMBER)
     I = 2          F = 1          (PRIME NUMBER)
     I = 3          F = 2          (PRIME NUMBER)
     I = 4          F = 3          (PRIME NUMBER)
     I = 5          F = 5          (PRIME NUMBER)
     I = 6          F = 8
     I = 7          F = 13         (PRIME NUMBER)
     I = 8          F = 21
     I = 9          F = 34
     I = 10         F = 55
     I = 11         F = 89         (PRIME NUMBER)
     I = 12         F = 144
     I = 13         F = 233        (PRIME NUMBER)
     I = 14         F = 377
     I = 15         F = 610
     I = 16         F = 987
     I = 17         F= 1597        (PRIME NUMBER)
     I = 18         F = 2584
     I = 19         F = 4181
     I = 20         F = 6765
     I = 21         F = 10946
     I = 22         F = 17711
     I = 23         F = 28657      (PRIME NUMBER)
     I = 24         F = 46368
```

EXERCISES

1. Prove that if $(a, b) = 1$ then

   (*i*) $(a^n, b^k) = 1$ for all $n \geq 1, k \geq 1$

   (*ii*) $(a+b, a^2 - ab + b^2)$ is either 1 or 3.

   (*iii*) $(a + b, a - b)$ is either 1 or 2.

2. Prove that $19 \mid 2^{2^{6k+2}} + 3$, for $k = 0, 1, 2, \ldots$.

3. Find all integers $n > 1$ such that $1^n + 2^n + \ldots\ldots + (n-1)^n$ is divisible by n.

4. Prove that for positive integer n

   (*i*) $n^2 \mid (n+1)^n - 1$ (*ii*) $(2^n - 1)^2 \mid 2(2^{(2^n-1)^n} - 1)$

5. Prove that for odd n, $n \mid 2^{n!} - 1$.

6. Prove that $(n, 2^{2^n} + 1) = 1$ for $n = 1, 2, \ldots\ldots$

7. Find all primes which can be represented both as sums and as differences of two primes.

8. Find four solutions of the equation $p^2 + 1 = q^2 + r^2$ with primes p, q and r.

9. Find the least positive integer n for which $n^4 + (n+1)^4$ is composite.

10. Show that for $n > 1$ the number (*i*) $1/5\ (2^{4n+2} + 1)$ is composite.

    (*ii*) $1/3\ (2^{2^{n+1}} + 2^{2^n} + 1)$ is composite .

11. Find all numbers p such that all six number p, p+2, p+6, p+8 and p+14 are primes.

12. Prove that the Mersenne number M_{29} is composite .

13. Find all positive integer $n > 1$ for which $(n-1)! + 1 = n^2$.

14. Prove that

    (*a*) $8 \mid 5^{2n} + 7$ [Hint: $5^{2(k+1)} + 7 = 5^2\ (5^{2k} + 7) + (7 - 5^2\ .\ 7)$]

    (*b*) $15 \mid 2^{4n} - 1$

    (*c*) $5 \mid 3^{3n+1} + 2^{n+1}$

    (*d*) $21 \mid 4^{n+1} + 5^{2n-1}$

    (*e*) $24 \mid 2.7^n + 3.5^n - 5$.

15. Prove that $(2n)! / n!\,(n+1)!$ is an integer.

16. Prove that if $d \mid n$ then $2^d - 1 \mid 2^n - 1$.

17. Show that $a^n \mid b^n$ implies $a \mid b$.

18. Show that the Fermat number F_5 is divisible by 641.

19. For $n \geq 2$, show that the last digit of the Fermat number $F_n = 2^{2^n} + 1$ is 7.

20. If n is a perfect number prove that $\sum_{d \mid n} 1/d = 2$.

21. Prove that every even perfect number is a triangular number.

22. If m and n are amicable numbers prove that

$$\left(\sum_{d \mid m} 1/d\right)^{-1} + \left(\sum_{d \mid n} 1/d\right)^{-1} = 1$$

CHAPTER
2

ARITHEMETICAL FUNCTIONS

2.0. INTRODUCTION :

Number theory like many other branches of mathematics, concern with sequence of real or complex numbers.

DEFINITION : A real or complex valued function defined on the positive integer is called an arithmetical function or a number theoretic function. Symbolically we write arithmetical function as

$$f : Z_+ \to R\ (C).$$

2.1. MOBIUS FUNCTION $\mu(n)$.

DEFINITION : The Mobius function μ is defined as follows:

$\mu(1) = 1;$

If $n > 1$, write $n = p_1^{a_1} \ldots . p_k^{a_k}$. Then

$\mu(n) = (-1)^k$ if $a_1 = a_2 = \ldots .. = a_k = 1$

$\mu(n) = 0$ otherwise ,

i.e. $\mu(n) = 0$ if and only if n has a square factor >1. In other words

$\mu = N \to \{-1, 0, 1\}$.

n:	1	2	3	4	5	6	7	8	9	10
$\mu(n)$:	1	–1	–1	0	–1	1	–1	0	0	1

Example 1. Find the value of $\mu(720)$.

Solution : $720 = 2^4 \times 3^2 \times 5$.

Hence $\mu(720) = 0$.

Example 2. Find $\sum_{k=1}^{n} \mu(k!)$.

Solution : $\sum_{k=1}^{n} \mu(k!) = \mu(1) + \mu(2!) + \mu(3!) + \mu(4!) + \mu(5!) + \ldots.$

$$= \mu(1) + \mu(2) + \mu(6) + \mu(24) + \ldots\ldots$$

$$\sum_{k=1}^{n} \mu(k!) \quad = 1 - 1 + 1 + 0 + 0 + \ldots. = 1.$$

Mobius function arises in many different places in number theory. We will study its most fundamental properties.

THEOREM 1 : If n ≥1 we have

$$\sum_{d|n} \mu(d) = I(n) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 0 \end{cases}$$

where I(n) is identity function.

PROOF : The formula is clearly true if n =1. Assume, then, that n > ! and write $n = \prod_{t-1}^{r} p_t^{a'}$ In the sum $\sum_{d|n} \mu(d)$ the only non zero terms come from d =1 and from those divisor of n which are product of distinct primes. Thus

$$\sum_{d|n} \mu(d) = \mu(1) + \mu(p_1) + \ldots.\mu(p_r) + \mu(p_1p_2) + \ldots.. \mu(p_{r\ 1}\ p_r)$$

$$+\ldots\ldots+ \mu(p_1\ p_2\ldots\ldots p_r)$$

$$\Rightarrow \sum_{d|n} \mu(d) = 1 + \binom{r}{2}(1) + \binom{r}{2}(-1)^2 + \ldots\ldots + \binom{r}{r}(-1)^r$$

$$= (1-1)^r = 0.$$

DEFINITION : For each positive integer n ,

we denote d(n) as

$$d(n) = \#\{k : k|n\} = \sum_{d|n} 1.$$

We also write r(n) for d(n).

σ(n) = sum of the divisors of n,

$\sigma_1(n)\quad \sum_{d|n} d^k$ = sum of the k^{th} power of divisor of n.

2.2. THE EULER TOTIENT FUNCTION $\Phi(n)$.

DEFINITION : If $n \geq 1$ the Euler totient $\varphi(n)$ is defined to be the number of positive integers not exceeding n which are relatively prime to n; thus

(1) $\varphi(n) = \sum_{d|n} 1$= Total number of integers less than n and relativity, prime to n $(k, n) = 1$.

TABLE 2

n	1	2	3	4	5	6	7	8	9	10
$\varphi(n)$	1	1	2	2	4	2	6	4	6	4

If $n = p$ (a prime) then $\varphi(n) = \varphi(p) = p-1$.

For example $\varphi(8) = 4$, because 1, 3, 5, 7 are relativity prime to 4.

Now the question is how to find $\varphi(n)$ when n is given in the standard form i.e.

$n = \prod_{i=1}^{r} p_i^{a_i}$. Like $\sum_{d|n} \mu(d)$ we will find the value of the summatory function of $\varphi(n)$ i.e. $\sum_{d|n} \varphi(d)$. The following theorem gives the answer to our question.

Before going to the theorem we will first define multiplicative function.

DEFINITION : An arithmetical function 'f ' is called multiplicative if f = 0 and if $f(m\,n) = f(m)\,f(n)$ for m, n; such that $(m,n)=1$.

A multiplicative function 'f ' is called completely multiplicative if we also have

$f(m\,n) = f(m)\,f(n)$ for all m, n.

A function f which is completely multiplicative is multiplicative but the converse is not true.

Example1 : Let $f_\alpha(n) = n^\alpha$, where α is a fixed real or complex number.

f_α is completely multiplicative because $(mn)^\alpha = m^\alpha . n^\alpha$ for all m, n . Hence $f_\alpha(n)$ is multiplicative.

Example 2 : The function $\mu(n)$ is multiplicative but not completely multiplicative. Consider first m, n such that (m, n) =1. Either m or n has a square factor. Hence $\mu(m\,n) = \mu(m)\,\mu(n) = 0$. If neither has a square factor, write $m = p_1\,p_2 \ldots p_r$, $n = q_1\,q_2 \ldots q_k$, $\mu\,(m\,n) = (-1)^{r+k}$, $\mu(m) = (-1)^r$, $\mu(n) = (-1)^k$ Since $(-1)^{k+r} = (-1)^r\,(-1)^k$, we have $\mu(m\,n) = (-1)^{r+k} = (-1)^r\,(-1)^k = \mu(m)\,\mu(n)$. Hence μ is multiplicative. But $\mu(n)$ is not completely multiplicative since $\mu\,(8) = 0$, $8 = 2^3$, $\mu(2) = -1$, $\mu(2^3) = (-1)\,(-1)\,(-1) = -1$. $\mu(8) \neq \mu(2)\,\mu(2)\,\mu(2)$.

Example 3 : Find φ (720).

$$720 = 2^4 \times 3^2 \times 5\ ,\ \varphi(720) = 720 \times (1-1/2)(1-1/3)\,(1-1/5)$$

$$= 12 \times 2 \times 4 = 96.$$

THEOREM 2 : If p is a prime and k > 0, then $\varphi\,(p^k) = p^k - p^{k-1} = p^k(1-1/p)$.

PROOF : There are p^{k-1} integer between 1 and p^k i. e.

$p, 2p, 3p\ldots..(p^{k-1})p$ which are divisible by p. The set $\{1,2,\ldots p^k\}$ contains exactly $p^k - p^{k-1}$ integer which are relatively prime to p^k and so, by definition of Eluler's φ function $\varphi\,(p^k) = p^k - p^{k-1}$.

For example

$$\varphi(8) = \varphi\,(2^3) = 2^3 - 2^2 = 8 - 4 = 4.$$

$$\varphi(27) = \varphi\,(3^3) = 3^3 - 3^2 = 27 - 9 = 18.$$

LEMMA : gcd (a,bc) = 1 if ar l only if gcd (a, b) = 1 and gcd (a,c) = 1, a, b, c are integers.

PROOF : Let d = gcd (a,b) then by definition of gcd d,a and d|b whence d|a and d|bc. This implies gcd (a,bc) $\geq$ d which forces d =1. Similarly for (a.c) = 1. Only if part: Let gcd (a,b) = 1 = gcd (a, c) and assume that gcd (a, bc) = $d_1 > 1$. Since d_1| bc, it follows p | bc; p is a prime divisor of d_1. Hence p | b or p | c. If p | b then gcd (a,b) $\geq$ p, a contradiction. Similar argument follows for gcd (a, c).

Thus d_1=1.

Example 4 : The function $\varphi(n)$ is a multiplicative function but not completely multiplicative function. $\varphi(n)$ is not completely multiplicative follows from the fact that

$\varphi(4) = 2$, $\varphi(2) = 1$, $\varphi(4) \neq \varphi(2)\,\varphi(2)$.

Multiplicity of $\varphi(n)$ will be proved in the next chapter on congruence .

THEOREM 3 : If the integer n>1 has the prime factorization $n = \prod_{i=1}^{r} p_i^{ai}$

then $\varphi(n) = n \prod_{i=1}^{r}\left(1-\frac{1}{p_i}\right)$.

PROOF : If $n = p^k$, there are p^k-1 positive numbers less then p^k, of which $p^{k-1}-1$ are multiples of p and the remainder prime to p.

Hence

$\varphi(p^k) = (p^k-1) -(p^{k-1}-1) = p^k (1 - 1/p)$

and the general value of φ(n) follow from the multiplicity of φ(n) .

$$\text{Since } n = \prod_{i=1}^{r} p_i^{a_i},\quad \varphi(n) = \prod_{i=1}^{r} \varphi(p_i^{a_i})$$

$$= \prod_{i=1}^{r} p_i^{a_i}(1 - 1/p_i)$$

$$= \prod_{i=1}^{r} p_i^{a_i} \prod_{i=1}^{r} (1-1/p_i)$$

$$= n \prod_{i=1}^{r} (1-1/p_i) = n \prod_{p|n} (1-1/p).$$

The right hand side product is independent of prime power of n.

THEOREM 4 : $\sum_{d|n} \varphi(d) = n$.

PROOF : If $n = \prod_{p|n} p^k$, then the divisor of n are the numbers $d = \prod_{p|n} p^{k'}$ where $0 \leq k \leq k$ for each p; and

$$\sum_{d|n} \varphi(d) = \sum_{p,\,k^1} \Pi\varphi(p^{k^1})$$

$= \prod_{p} \{1+ \varphi(p) + \varphi(p^2) + \ldots + \varphi(p^k)\}$ by the multiplicatve property of φ(n) .

But $1 + \varphi(p) + \varphi(p^2) +\ldots+ \varphi(p^k)$

$$= 1+ (p-1) + p(p-1) +\ldots+p^{k-1}(p-1)$$

$$= p^k.$$

So that

$$\sum_{d|n} \varphi(d) = \prod_{p|n} p^k = n.$$

We can prove theorem 4 without using multiplicative property.

Second method of proof of theorem 4 : Let 'S' denote the set $\{1, 2,..n\}$. We distribute the integers of S into disjoint set as follows: For each divisor d of n, let

$$A\,(d) = \{k: (k, n) = d,\ 1 \le k \le n\}.$$

$$\bigcup_{d} A\,(d) = S.$$

If f (d) denotes the number of integer in $A(d)$ we have

$$\sum_{d|n} f(d) = n.$$

But $(k, n) = d$ if and only if $(k/d, n/d) = 1$,

and $0< k \le n$ if and only if $0 < k/d \le n/d$.

Put $q = k/d$ then $0 < q \le n/d$. There is one–to–one correspondence between the elements in $A(d)$ and those integers q such that $(q, n/d) = 1$. The number of such q is $\varphi(n/d)$. Hence $f(d) = \varphi(n/d)$. We have

$$\sum_{d|n} \varphi(n/d) = n.$$

This implies $\sum_{d|n} \varphi(d) = n$, because if d runs through divisor of n so does n/d.

This completes the proof.

RELATION BETWEEN Φ(n) AND μ(n).

THEOREM 5 : If $n \ge 1$, we have $\varphi(n) = \sum_{d|n} \mu(d)\, n/d$.

PROOF : $\varphi(n) = \sum_{\substack{k=1 \\ (n,k)=1}}^{n} 1 = \sum_{k=1}^{n} [1/(n,k)]$

where [x] is the greatest integer function.

By Theorem 1.

$$\varphi(n) = \sum_{k=1}^{n} \sum_{d|(n,k)} \mu(d) = \sum_{\substack{d|n \\ d|k}} \mu(d).$$

For a fixed divisor d of n we sum over all those k in the range $1 \le k \le n$ which are multiples of d .

Taking k = q d, $1 \le k < n$ if and only if $1 \le q \le n/d$. Hence

$$\varphi(n) = \sum_{d|n} \sum_{q=1}^{n/d} \mu(d) = \sum_{d|n} \mu(d) \sum_{q=1}^{n/d} 1$$

$$= \sum_{d|n} \mu(d)\, n/d.$$

Remark. Theorem 3 can be deduced from theorem 5 by using the fact that

$$\prod_{p|n} (1 - 1/p) = \prod_{i=1}^{r} (1 - 1/p_i)$$

$$= 1 - \Sigma\, 1/p_i + \Sigma\, 1/p_ip_j - - - - + (-1)^r / (p_1 + p_2 - - - + p_4)$$

$$= \sum_{d|n} \mu(d)/d.$$

Hence $\varphi(n) = \sum_{d|n} \mu(d)\, n/d = n \prod_{d|n} (1 - 1/p)$.

THEOREM 6 : If d = (m, n), then $\varphi(m\, n) = \varphi(m)\, \varphi(n)\, (d/\varphi(d))$.

PROOF : $\varphi(n)/n = \prod_{d|n} (1-1/p)$, $\varphi(m) / m = \prod_{p|m} (1 - 1/p)$.

Since every prime divisor of mn is either a divisor of m or n, and those primes which divide both m and n also divide (m, n), we have

$$\varphi(mn) / mn = \prod_{p|mn} (1 - 1/p) = \frac{\prod_{p|m} (1-1/p) \prod_{p|n} (1-1/p)}{\prod_{p|(m,n)} (1-1/p)}$$

$$= \frac{\phi(m)/m . \phi(n)/n}{\phi(d)/n}$$

$$\Rightarrow \qquad \varphi(mn) = \varphi(m)\, \varphi(n)\, (d/\varphi(d)).$$

Corollary : $\varphi(mn) = \varphi(m)\varphi(n)$ if (m,n) =1. It is trivial by putting d = 1 in Theorem 6.

THEOREM 7 : $\varphi(n)$ is even for n> 3. If n has r distinct odd prime factors then $2^r|\varphi(n)$.

PROOF : If $n = 2^r$, $r \ge 2$ $\varphi(n) = \varphi(2^r) = 2^r - 2^{r-1}$ implies that $\varphi(n)$ is even for $n \ge 3$. If n has at least one odd prime factor we write

$$\varphi(n) = n \prod_{p\ n} (p-1/p)$$

$$= \frac{n}{\prod p} \prod_{p|n} (p-1)$$

$$= k \prod_{p|n} (p-1),$$

where $k = n/ \prod_{p|n} p$ is an integer and since p is odd, p–1 is even. If n has r distinct odd prime factor then $2^r | \varphi(n)$.

Remark : All φ(n) are even. Only odd φ(n) are for n = 2 and n = 1.

Example : Evaluate φ(350).

As $350 = 2 \times 5^2 \times 7$

$\varphi(350) = \varphi(2)\,\varphi(5^2)\ \ \varphi(7) = 1 \times (5^2-5).6 = 120.$

THEOREM 8 : Let m and n be integer both greater than 1 and every prime divisor of n is a prime divisor of m. Then

(i) $\varphi(m\,n) = n\,\varphi(m)$

(ii) $\varphi(n^2) = n\,\varphi(n)$ for all $n \geq 1$.

PROOF : *(i)* By hypothesis m and n can be written as

$$n = p_1^{r_1} p_2^{r_2} \ldots . p_k^{r_k}$$

and $$m = p_1^{S_1} p_2^{S_2} \ldots\ p_k^{S_k} p_{k+1}^{S_{k+1}} \ldots p_t^{S_t},$$

p_i being prime, r_i, s_j integers each ≥ 1 and t ≥ k.

$$\begin{aligned}
\varphi(m\,n) &= \varphi(p_1^{S_1+r_1} p_2^{S_2+r_2} \ldots\ p_k^{S_k+r_k} p_{k+1}^{S_{k+1}} \ldots p_t^{S_t}) \\
&= \varphi\ \ (p_1^{S_1+r_1}) \ldots\ldots\ \varphi(p_k^{S_k+r_k}) \ldots\ \ \varphi(p_t^{S_t}) \\
&= (p_1^{S_1+r_1} - p_1^{S_1+r_1-1})(p_2^{S_2+r_2} - p_2^{S_2+r_2-1}) . \\
&\quad \varphi(p_{k+1}^{S_{k+1}}) \ldots \varphi(p_t^{S_t}) \\
&= (p_1^{r_1} \ldots\ p_k^{r_k}\ (p_1^{S_1} - p_1^{S_1-1})(p_2^{S_2} - p_2^{S_2-1}) . \\
&\quad \varphi(p_{k+1}^{S_{k+1}}) \ldots \varphi(p_t^{S_t}) \\
&= m\,.\varphi(p_1^{S_1}) \ldots\ldots \varphi(p_t^{S_t}) \\
&= m\,\varphi(n).
\end{aligned}$$

Putting m = n we get *(ii)*

But if the same prime divides both m and n then n φ (m) = mφ(n) is not always true. For example taking m = 6, n = 8, 8 · φ (6) ≠ 6 · φ(8)

Example : If (m, n) = 1 then (φ(m),φ(n)) = 1 is not always true. For example take m = 6, n = 7, (6,7) = 1. But φ(m) = 2, φ(n) = 6, (φ(m),φ(n)) >1.

THEOREM 9 : For n > 1, the sum of the positive integer is less than n and relatively prime to n is ½ nφ(n).

PROOF : Let $a_1\ a_2 \,..\, a\varphi_{(n)}$ be the positive integers less than n and relatively prime to n. Now since gcd (a,n) = 1 if and only if gcd (n–a, n) = 1 , we have

$$\begin{aligned} a_1 + a_2 + \ldots + a\varphi_{(n)} & \\ &= (n-a_1) + (n-a_2) + \ldots. + (n-a\varphi_n) \\ &= \varphi(n)\, n - (a_1 + a_2 + \ldots + a\varphi_{(n)}) \end{aligned}$$

Hence, $2(a_1 + a_2 + \ldots.. + a\varphi_{(n)}) = \varphi(n)\, n$

$\Rightarrow a_1 + a_2 + \ldots. + a\varphi_{(n)} = n\, \varphi(n) / 2.$

Example : $\sum_{\substack{1\le k\le 20 \\ (k,20)=1}} k = 1 + 3 + 7 + 11 + 13 + 17 + 9 + 19.$

$= 80 \quad = 20\varphi(20) / 2.$

$\varphi(20) = \varphi\,(2^2,5) = \varphi(2^2)\, \varphi(5) = 2 \times 4 = 8.$

THEOREM 10 : Let f and F be number theoretic function such that

$$F\,(n) = \sum_{d|n} f(d).$$

Then , for any positive integer N,

$$\sum_{n=1}^{N} F(n) = \sum_{k=1}^{N} f(k)\, [N/k],$$

PROOF : We note that

$$(1) \qquad \sum_{n=1}^{N} F(n) = \sum_{k=1}^{N} \sum_{d|n} f(d)$$

For a fixed positive integer k ≤ N, the term f(k) appears in $\sum_{d|n} f(d)$ if and only if k is a divisor of n. There are exactly [N/k] number of terms among 1,2 …N which are divisible by k. They are k, 2k, 3k … [N/k] k. Thus for each k such that 1 ≤ k ≤ N, f(k) is a term of the sum $\sum_{d|n} f(d)$ for [N/k] different positive integer less than or equal to N. We may write the double sum in (1) as

$$\sum_{n=1}^{N} \sum_{d|n} f(d) = \sum_{n=1}^{N} f(k)\, [N/k].$$

Example 1 : $\sum_{d=1}^{n} \varphi(d)\, [n/d] = n(n+1)/2$ for any positive integer n

Solution : By Theorem 9,

$$\sum_{d=1}^{n} \varphi(d)[n/d] = \sum_{k=1}^{n} \sum_{d|k} \varphi(d)$$

$$= \sum_{k=1}^{n} k = 1 + 2 + .. + n = n(n+1)/2.$$

Example 2 : Prove that $n/\varphi(n) = \sum_{d|n} \mu^2(d) / \varphi(d)$.

Solution : If $n = p_1 p_2 \ldots p_r$ (We consider the product of distinct primes only because $\mu(n) = 0$, if n contains a square .

Again $\mu^2(n) / \varphi(n)$ is multiplicative.

Let $$G(n) = \sum_{d|n} \mu^2(d) / \varphi(d)$$

If $$n = \sum_{i=1}^{r} p_i^{ai}$$

then $$G(n) = \sum_{i=1}^{r} G(pi^{ai})$$

$$= (1 + 1/\varphi(p_1))(1 + 1/\varphi(p_2)).....(1 + 1/\varphi(p_r))$$
$$= (p_1/p_1-1)(p_2/p_2-1).....(p_r/p_r-1)$$
$$= 1/(1-1/p_1)\, 1/(1-1/p_2)......1/(1-1/p_r)$$
$$= n/\varphi(n).$$

Example 3 : Find all integers n such that

(i) $\varphi(n) = \varphi(2n)$

(ii) $\varphi(n) = n/2$

(iii) $\varphi(n) = 12$

Solution : Proof of *(i)* $\varphi(n) = \varphi(2n)$ holds for $n = 1, 3, 5, 7, \ldots.$

(ii) $\varphi(n) = n/2$ if and only if $n = 2^k$ for same $k \geq 1$.

If $n = 2^k$, $\varphi(2^k) = 2^k - 2^{k-1} = 2^{k-1} = n / 2$

If $n = 2^k N$, N is odd then

$$\varphi(n) = \varphi(2^k N) = 2^k N / 2$$

$\Rightarrow$ $\varphi(2^k)\varphi(N) = 2^{k-1} N.$

$\Rightarrow$ $2^{k-1}\varphi(N) = 2^{k-1} N.$

$\Rightarrow$ $N = 1.$

(iii)$12 = 3 \times 4$. Suppose $n = st$, $\varphi(st) = \varphi(s)\,\varphi(t)$

To show that $\varphi(s) = 3$, $\varphi(t) = 4$.

There is no s for which $\varphi(s) = 3$.

$$12 = 2 \times 6, \quad \varphi(3) = 2, \quad \varphi(7) = 6.$$
$$\varphi(4) = 2$$
$$12 = 12 \times 1 \quad \varphi(6) = 2$$
$$\varphi(13) = 12, \varphi(2) = 1, \quad \varphi(1) = 1$$

The number are $n =$ 13, 21, 26, 28, 42.

Example 4 : Find all solutions of (*i*) $\varphi(n) = 24$ and (*ii*) $\varphi(n) = 16$

Solution : *(i)* $\varphi(n) = 24$

(ii) $\varphi(n) = 16$

We factorize $24 = 3 \times 8$

$$= 2 \times 12$$
$$= 6 \times 4$$

In the first case there is no value of n for which $\varphi(n)$ is 3 hence we discard the first factor. Consider the second and third factors.

$\varphi(3) = 2$ $\quad\varphi(13) = 12$ $\quad\varphi(5) = 4$ $\quad\varphi(12) = 4$

$\varphi(4) = 2$ $\quad\varphi(7) = 6$ $\quad\varphi(9) = 6$

$\varphi(6) = 2$ $\quad\varphi(10) = 4$ $\quad\varphi(8) = 4$

So the integer n for which

$\varphi(n) = 24$ is given by 35, 39, 45, 52, 56, 70, 72, 78, 84, 90.

Verification : Since $(3,13) = 1$

$$\varphi(39) = \varphi(3\text{x}13) = \varphi(3) \times \varphi(13) = 2 \times 12 = 24$$
$$(4,13) = 1 \Rightarrow \varphi(52) = \varphi(4 \times 13) = \varphi(4)\,\varphi(13) = 2 \times 12 = 24$$
$$(6,13) = 1 \Rightarrow \varphi(78) = \varphi(6 \times 13) = \varphi(6)\,\varphi(13) = 2 \times 12 = 24$$
$$(5,7) = 1 \Rightarrow \varphi(35) = \varphi(5 \times 7) = \varphi(5)\ \varphi(7) = 4 \times 6 = 24$$
$$(7,10) = 1 \Rightarrow \varphi(70) = \varphi(7 \times 10) = \varphi(7)\ \varphi(10) = 6 \times 4 = 24.$$

(ii) $\varphi(n) = 16 = 2 \times 8 = 16 \times 1 = 4 \times 4$

$\varphi(3) = 2$ $\quad\varphi(5) = 4$ $\quad\varphi(17) = 16$

$\varphi(4) = 2$ $\quad\varphi(8) = 4$ $\quad\varphi(2) = 1$

$\varphi(6) = 2$ $\quad\varphi(10) = 4$

$\varphi(1) = 1$

$\varphi(16) = 8$ $\quad\varphi(15) = 8$

So the solution of $\varphi(n) = 16$ are

$$n = 17, 32, 34, 40, 48, 60 .$$

Example 5 : There are infinitely many integers n for which (i) $10 \mid \varphi(n)$ and *(ii)* $\varphi(n)$ is a perfect square.

Solution : (i) $\varphi(11^k) = 11^k - 11^{k-1} = 11^{k-1}. 10$

Hence $10 \mid \varphi(n)$ where $n = 11^k$, $k = 1, 2, 3....$

(ii) If $n = 2^{2k+1}$, $k = 1, 2, 3, 4....$

Then $\varphi(2^{2k+1}) = 2^{2k+1} - 2^{2k} = 2^{2k}(2-1) = 2^{2k} = (2^k)^2$

Hence for $n= 2^{2k+1}$, $k = 1, 2, 3, 4, \ldots$ $\varphi(n)$ is a perfect square.

2.3. ARITHMETICAL FUNCTION d(n) AND σ(n).

DEFINITION 1 : The number of positive divisors of n is called as divisor function and is denoted as d (n).

$$d(n) = \sum_{d|n} 1.$$

DEFINITIONS 2 : The sum of the positive divisors of n is denoted as $\sigma(n)$. Symbolically we write

$$\sigma(n) = \sum_{d|n} d.$$

We define $\sigma_\alpha(n)$ as

$$\sigma_\alpha(n) = \sum_{d|n} d^\alpha \qquad \alpha \geq 0$$

If $\alpha = 0$, $\sigma_o(n) = d(n)$.

If $\alpha = 1$, $\sigma_1(n) = \sigma(n)$.

Example : $n = 6$, $d(n) = 4$,

$$\sigma(n) = 1 + 2 + 3 + 6 = 12$$

If n is a prime p then $d(n) = 2$, and $\sigma(n) = 1 + p$, To find formula for $\sigma(n)$ when n is in the standard form.

THEOREM 11 : If $n = p_1^{a_1} \ldots\ldots p_r^{a_r}$, then

$$d(n) = (a_1+1)(a_2+1) \ldots\ldots (a_r+1) \text{ and}$$

$$\sigma(n) = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

PROOF : If $n = p^a$ the divisors are

$1, p, p^2, \ldots p^a$ i.e. there are a+1 number of divisors of p^a.

Hence $d(p^a) = a + 1$

and $\sigma(n) = \sigma(p^a) = 1 + p + p^2 + \ldots p^a = p^{a+1} - 1/(p-1).$

Assume that the theorem is true whenever n has k or fewer distinct prime factors.

Let $n = n' p^a$ where n' has k distinct prime factor and p, which is prime not a factor of n'

If $n' = p_1^{a_1} p_2^{a_2} \ldots\ldots p_k^{a_k}$

then $d(n') = (a_1+1)(a_2+1)\ldots(a_k+1)$

and $$\sigma(n') = \prod_{i=1}^{r} \frac{p_i^{a_i+1} - 1}{p_i - 1}.$$

Let $d_1, d_2, \ldots d_s$ denote the d(n') divisor of n'. Then the divisor of n are $d_1, d_2, \ldots d_s, pd_1, pd_2 \ldots p\, d_s, p^2d_1, p^2d_2 \ldots\ldots, p^ad, p^ad_2 \ldots p^ad_s$.

Thus

$$\begin{aligned} d(n) &= d(n')(a+1) \\ &= (a_1 + 1)(a_2 + 1) \ldots (a_k + 1)(a + 1) \end{aligned}$$

and similarly

$$\begin{aligned} \sigma(n) &= \sigma(n') + p\,\sigma(n') + \ldots + p^a\sigma(n') \\ &= \sigma(n')(1 + p + \ldots + p^a) \\ &= \sigma(n')\left(\frac{p^{a+1} - 1}{p-1}\right) \\ &= \sum_{i=1}^{k}\left(\frac{p_i^{a_i+1} - 1}{p_i - 1}\right)(p^{a+1} - 1/p-1). \end{aligned}$$

Our theorem follows by mathematical induction.

Corollary 1 : If $n = p_1^{a_1} p_2^{a_2} \ldots\ldots p_s^{a_s}$

then $d(n) = d(p_1^{a_1})\, d(p_2^{a_2}) \ldots\ldots d(p_s^{a_s})$

$\sigma(n) = \sigma(p_1^{a_1})\, \sigma(p_2^{a_2}) \ldots\ldots \sigma(p_s^{a_s})$

That is d(n) and σ(n) are multiplicative functions.

Example 1 : $d(100) = d(2^2 \times 5^2) = 3 \times 3 = 9.$

$\sigma(100) = ((2^3-1)/(2-1))((5^3-1)/(5/1)) = 7 \times 31 = 217.$

Example 2 : $\prod_{d|n} d = n^{d(n)/2}$.

Solution : If $n= p^{\alpha}$, divisors of n are $1, p, p^2 \ldots . p^{\alpha}$

Hence $\prod_{d|p^{\alpha}} d = 1, p, p^2 \ldots p^{\alpha} = p^{\alpha(\alpha+1)/2} = (p^{\alpha})^{(\alpha+1)/2} = (p^{\alpha})^{d(p^{\alpha})/2}$

In this case $\prod_{d|n} d = n^{d(n)/2}$.

If $n= \prod_{p|n} p^{\alpha}$, $d(n) = \prod_{p|n} d(p^{\alpha})$.

Hence $\prod_{d|n} d = \prod_{p|n}(\prod_{d|p^{\alpha}}) = \{\prod_{p|n}(p^{\alpha})\} / 2 = n^{d(n)/2}$.

Second solution : $d \mid n \Rightarrow n = d\, d'$

$\Rightarrow$ $d' | n$ and $d' = n/d$. The divisor d of n are in pairs (d, n/d)

$\Rightarrow$(product of all divisor of n)^2 = $(\prod_{d|n}(d))^2 = n^{d(n)}$

$\Rightarrow$ $\prod_{d|n} (d) = n^{d(n)/2}$.

2.4. GENERALIZED EULER'S TOTIENT FUNCTION.

We generalize Euler's totient function $\varphi(n)$ to $\varphi(x, n)$ as follow:

Let x be a positive real number and for all n let $\varphi(x, n)$ = the number of integers y such that $1 \le y \le x$ and $(y, n) = 1$.

Thus $\varphi(n,n) = \varphi(n)$. For all x and n,

$$\varphi(x,n) = \Sigma\, \mu(d)\, [x/d] .$$

Also we can extend $\varphi(n)$ to $\varphi(*,k)$ as follows $\varphi(n,k)$ = the number of integer x such that $1 \le x \le n$ and $(x, n) = (n + k - x, n) = 1$

$$\varphi(*, n) = \varphi.$$

If $(m, n) = 1$, then

$$\varphi(mn, k) = \varphi(m, k)\, \varphi(n, k)$$

i. e. $\varphi(*, k)$ is a multiplicative function.

THEOREM 12 : $\varphi(n, k) = n \prod_{p|n} \left(1 - \frac{\varepsilon(p)}{p}\right)$.

where $\varepsilon(p) = \begin{cases} 1 \text{ if } p|k \\ 2 \text{ if } p \nmid k. \end{cases}$

The proof is difficult.

2.5. LIOUVILLE'S FUNCTION λ(n).

We define another arithmetical function λ(n) called as Liouvilles function as follows .

DEFINITION : We have $\lambda(1) = 1$ and if $n = p_1^{a_1} p_2^{a_2} \ldots\ldots p_k^{a_k}$ we define

$$\lambda(n) = (-1)^{a_1+a_2+a_3+\ldots+ac}$$

λ(n) is completely multiplicative since if $n = p_1^{a_1} p_2^{a_2} \ldots\ldots p_k^{a_k}$;

$$\begin{aligned} m &= q_1^{b_1} q_2^{b2} \ldots\ldots q_s^{b_s} \\ \lambda(nm) &= (-1)^{a_1.a_2....a_k.b_1.b_2..b_s} \\ &= (-1)^{a_1.a_2...a_k} (-1)^{b_1.b_2...b_s} \\ &= \lambda(n)\,\lambda(m) \text{ for all } m,n. \end{aligned}$$

THEOREM 13 : For every $n \geq 1$ we have

$$\sum_{d|n} \lambda(d) = \begin{cases} 1 \text{ if n is a square} \\ 0 \text{ otherwise.} \end{cases}$$

PROOF : Let $g(n) = \sum_{d|n} \lambda(d)$. Then g is multiplicative, so to determine g(n) we need only to compute $g(p^a)$ for prime powers. We have

$$\begin{aligned} g(p^a) &= \sum_{d|p^\alpha} \lambda(d) = 1 + \lambda(p) + \lambda(p^2) + \ldots.. + \lambda(p^a) \\ &= 1 - 1 + 1 \ldots. + (-1)^a \\ &= \begin{cases} 0 \text{ if a is odd} \\ 1 \text{ if is even} \end{cases} \end{aligned}$$

Hence if $n = \prod_{i=1}^{r} p_i^{ai}$ we have

$g(n) = \prod_{i=1}^{r} g\ (p_i^{a_i})$. If any exponent a_i is odd then $g(p_i^{a_i}) = 0$ so $g(n) = 0$.

If all the exponents a_i are even then $g(p_i^{a_i}) = 1$ for all i and $g(n) = 1$. This proves that $g(n) = 1$ if n is a square and $g(n) = 0$ otherwise.

Hence proved.

2.6. VON–MANGOLDT FUNCTION

We define Mangoldt function Λ (n) as follows :

DEFINITION : For every integer $n \geq 1$ we define

$$\Lambda(n) = \begin{cases} \log p, \text{if } n = p^m \text{ forsome prime and some } m \geq 1 \\ 0 \text{ otherwise.} \end{cases}$$

TABLE 1 : VALUES OF Λ(N)

n:	1	2	3	4	5	6	7	8	9	10
Λ(n):	0	log 2	log 3	log 2	log 5	0	log 7	log 2	log 3	0

We note that $\Lambda(2) = \Lambda(2^k)$

i.e. $\Lambda(p) = \Lambda(p^k)$, p is a prime and $k > 1$. It is important to find the summatory function $\sum\limits_{d|n} \Lambda(d)$.

THEOREM 14 : If $n \geq 1$ we have

(2) $$\sum_{d|n} \Lambda(d) = \log n.$$

PROOF : The theorem is trivial when n = 1 because $\Lambda(1) = 0 = \log 1$. Assume

$$n > 1 \text{ and write } n = \prod_{i=1}^{r} p_i^{a_i}.$$

Taking logarithm of both sides we have

$$\log n = \prod_{i=1}^{r} a_i \log p_i.$$

Consider the L.H.S $\sum\limits_{d|n} \Lambda(d)$.

The only non–zero terms in the sum came from those divisors d of the form p_i^m for $m = 1, 2 \ldots a_i$ and $i = 1, 2, \ldots . r$.

Hence

$$\sum_{d|n} \Lambda(d) = \sum_{i=1}^{r} \sum_{m=1}^{a_i} \Lambda(p_i^m)$$

$$= \sum_{i=1}^{r} \sum_{m=1}^{a_i} \log p_i = \sum_{i=1}^{r} a_i \log p_i = \log n.$$

This proves (2).

THEOREM 15 : If $n \geq 1$ we have

$$\Lambda(n) = \sum_{d|n} \mu(d) \log n/d = -\sum_{d|n} \mu(d) \log d$$

PROOF : $\Lambda(n) = \sum_{d|n} \mu(d) \log n/d = \sum_{d|n} \mu(d) (\log n - \log d)$

$$= \log n \sum_{d|n} \mu(d) - \sum_{d|n} \mu(d) \log d$$

$$= -\sum_{d|n} \mu(d) \log d.$$

by theorem 1, as the first term is zero.

2.7. AVERAGES OF ARITHMETICAL FUNCTION.

We have observed that some arithmetical function behave most irregularly for large value of n. So we consider the average or mean of such arithmetical functions.

$$\overline{f(n)} = 1/n \sum_{k=1}^{n} f(k).$$

$\overline{f(n)}$ behave more regularly than f (n). Also we have

(3) $\lim_{n\to\infty} d(n) / \log n = 1.$

We call the average order of d(n) is log n. To study the average of an arbitrary function f(n) we need a knowledge of its partial sums $\sum_{k=1}^{n} f(k)$.

DEFINITION : If $g(x) > 0$ for all $x \geq a$,

we write $f(x) = 0(g(x))$

i.e. there exists a constant $M > 0$ such that $|f(x)| \leq M\ g(x)$ for all x ≥ a

DEFINITION : If $\lim\limits_{x\to\infty} f(x)/g(x)=1.$

We say that f(x) is asymptotic to g (x) as x → ∞,

and we write $f(x) \sim g(x)$ as $x \to \infty$.

The key theorem for all the theorems on the averages of arithmetical function is the following Euler's summation formula.

THEOREM 16 : If f has a continuous derivative f ′ on the interval [y, x], where $0 < y < x$, then

$$(4) \sum_{y<n\le x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)\,dt$$

$+ f(x)([x]-x)-f(y)([y]-y).$

PROOF : Let m = [y], k= [x]. For integer n and n–1 in [y, x] we have

$$\int_{n-1}^{n} [t]\, f'(t)\,dt = \int_{n-1}^{n} (n-1)\, f'(t)\,dt$$

$$= (n-1)\,\{f(n)-f(n-1\}$$
$$= \{\, n f(n) - (n-1)\, f(n-1)\} - f(n).$$

Summing from n = m + 1 to n = k we have

$$\sum_{n=m+1}^{k} \int_{n-1}^{n} [t]\, f'(t)\,dt = \sum_{n=m+1}^{k} \{n\, f(n)-(n-1)\, f(n+1)\} - \sum_{n=m+1}^{k} f(n)$$

$$\Rightarrow \qquad \int_m^k [t]\, f'(t)\,dt = kf(k) - mf(m) - \sum_{y<n\le x} f(n)$$

$$\Rightarrow \qquad \sum_{y<n\le x} f(n) = -\int_m^k [t]\, f'(t)\,dt + kf(k) - mf(m)$$

$$+ kf(x) - mf(y) - kf(x) + mf(y)$$

(5) $$\Rightarrow \quad \sum_{y<n\le x} f(n) = -\int_y^x [t]\, f'(t)\, dt + kf(x) - mf(y)$$

Again

(6) $$\int_y^x f(t)\, dt = xf(x) - yf(y) - \int_y^x tf'(t)\, dt$$

Subtracting (6) from (5) we have

$$\sum_{y<n\le x} f(n) - \int_y^x f(y)\, dt = -\int_y^x [t]\, f'(t)\, dt \;\; + kf(x) - mf(y)$$

$$- x\, f(x) + yf(y) + \int_y^x tf'(t)dt = \int_y^x (t-[t])\, f'(t)\, dt + f(x)\,([x]-x)$$

$$- f(y)\,([y]-y)$$

$$\Rightarrow \quad \sum_{y<n\le x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])\, f'(t)dt \; + f(x)\,([x]$$

$- f(y)\,([y] - y).$

THEOREM 17 : If $x \ge 1$ we have

(a) $\sum_{n\le x} 1/n = \log x + C + O\,(1/x)$. C is a constant.

(b) $\sum_{n\ge x} 1/n^s = 0\,(x^{1-s})$ if $s > 1$.

(c) $\sum_{n\ge x} n^{\alpha} = x^{\alpha+1}/\alpha+1 + O(x^{\alpha})$ if $\alpha \ge 0$.

PROOF : To prove *(a)*

take $f(t) = 1/t$ in theorem 16 to obtain

$$\sum_{n\le x} 1/n = \int_1^x dt/t - \int_1^x t-[t]/t^2\, dt \quad + 1 - (x-[x]/x)$$

$$= \log x - \int_1^x t-[t]/t^2 \ dt \ + 1 + 0(1/x)$$

$$= \log x + 1 - \int_1^\infty t-[t]/t^2\,dt + \int_1^\infty t-[t]/t^2\,dt + 0(1/x)$$

Now

$$0 \le \int_x^\infty t-[t]/t^2\,dt \ \le \int_x^\infty 1/t^2\,dt = 1/x$$

So we get

$$\sum_{n\le x} 1/n = \log x + 1 - \int_1^\infty t-[t]/t^2\,dt + 0(1/x)$$

$$= \log x + C + 0\,(1/x),$$

$$\text{if } C = 1 - \int_1^\infty t-[t]/t^2\,dt$$

(b) $\sum_{n>x} 1/n^s = \sum_{n=1}^{\infty} 1/n^s - \sum_{n\le x} 1/n^s = 0(x^{1-s})$ if $s > 1$.

(c) Taking $f(t) = t^\alpha$ we obtain

$$\sum_{n\le x} n^\alpha = \int_1^x t^\alpha\,dt + \alpha\int_1^x t^{\alpha-1}(t-[t])dt \ + 1 - (x-[x])\,x^\alpha$$

$$= x^{\alpha+1}/\alpha{+}1 - 1/\alpha{+}1 + 0(\alpha)\int_1^x t^{\alpha-1}dt) \ + 0(x^\alpha).$$

$$= x^{\alpha+1}/\alpha{+}1 + 0\,(x^\alpha).$$

THEOREM 18 : For all $x \ge 1$ we have

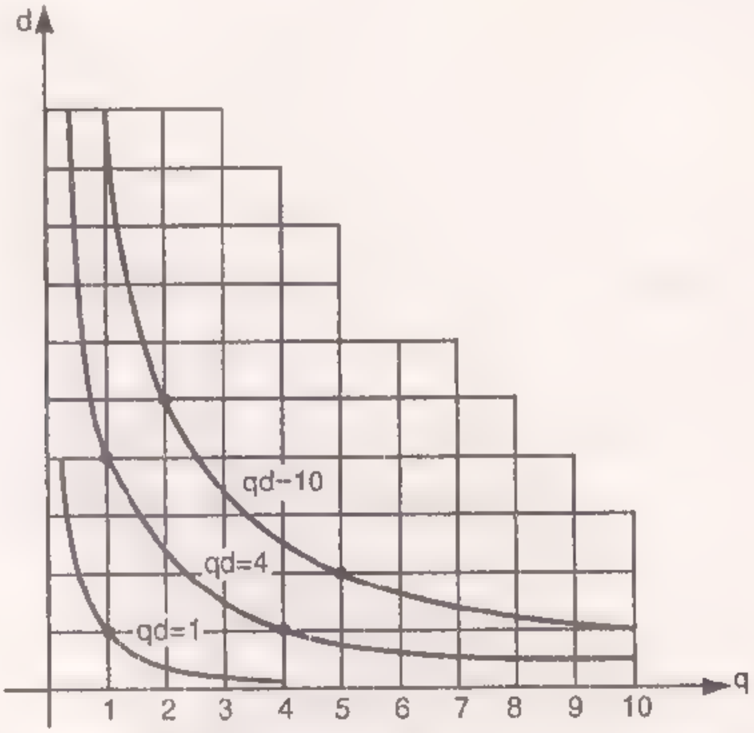
$$\sum_{n\le x} d(n) = x\log x + 0(x).$$

(b) $\sum_{n\le x} d(n) \sim x\log x$ as $x \to \infty$.

PROOF : Since $d(n) = \sum_{d|n} 1$ we have

$$(7)\ \sum_{n\le x} d(n) = \sum_{n\le x}\ \sum_{n\le x} 1 = \sum_{\substack{q,d\\ qd\le x}} 1.$$

(Since d|n we write n = q d with qd ≤ x)



The sum is extended over certain lattice point in the qd–plane (by lattice point we mean points with integer coordinates). The lattice points with qd = n lie on a hyperbola, so the sum in (7) counts the number of lattice points which lie on the hyperbola corresponding to n = 1, 2,…, [x]. For each fixed d ≤ x we can count first those lattice points on the horizontal line segment $1 \le q \le x/d$, and then sum over all d ≤ x . Then (7) becomes.

$$(8)\quad \sum_{n\le x} d(n) = \sum_{d\le x}\ \sum_{d\le x/d} 1$$

$$= \sum_{d\le x} [\, x/d + 0(1)]$$

(by *(c)* of Theorem 15)

$$= x \sum_{d\le x} 1/d + 0(x)$$

$$= x\,\{\log x + C + O\,(1/x)\} +)(x) \text{ (c is a constant)}$$

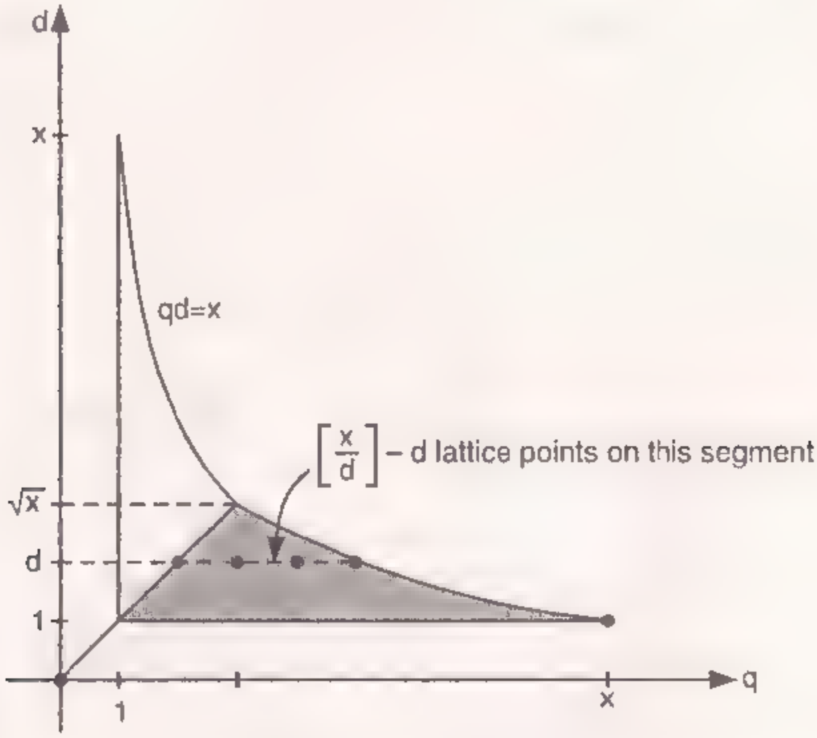
$$= x \log x + O\,(x).$$

That is $\sum_{n\le x} d(n) \sim x \log x$ as $x \to \infty$.

THEOREM 19 : For x ≥ 1

$$(9)\ \sum_{n\le x} d(n) = x \log x + (2C-1)\, x + O\,(\sqrt{x}).$$

PROOF :


d
x
√x
d
1
qd=x
$\left[\frac{x}{d}\right]$ – d lattice points on this segment
q
1
x

The total number of lattice points in the region about the line q = d is equal to twice the number below the q = d plus the number on the bisecting s line segments. Thus

$$\sum_{n\le x} d(n) = 2 \sum_{n\le\sqrt{x}} \{ [x/d]-d\} + [\sqrt{x}].$$

$$= 2 \sum_{d\le\sqrt{x}} \{ x/d + 0\,(1)-d\} + 0\,[\sqrt{x}]$$

$$(\text{Since } [x/d] = x/d + \{x/d\}$$

$$= x/d + O\,(1))$$

$$= 2x \sum_{d\le\sqrt{x}} 1/d + 0(1) \sum_{d\le\sqrt{x}} 1 - 2 \sum_{d\le\sqrt{x}} d + 0\,(\sqrt{x})$$

$$= 2x\,\{\log\sqrt{x} + C + O(1/\sqrt{x})\} + O(\sqrt{x}) - 2\,\{x/2 + O(\sqrt{x})\} + O(\sqrt{x})$$

$$= x \log x + (2C-1)\,x + O\,(\sqrt{x}).$$

This completes the proof of the theorem 19.

This formula (9) is due to Dirichlet and is an improvement over the formula (8)

THEOREM 20 : For $x > 1$ we have

$$(10) \qquad \sum_{n\le x} \varphi(n) = 3/\pi^2 + O(x \log x).$$

So the average order of $\varphi(n)$ is $3n / \pi^2$

PROOF : We know

$$\varphi(n) = \sum_{d|n} \mu(d)\, n/d$$

Hence

$$\sum_{n\le x} \varphi(n) = \sum_{n\le x} \sum_{d|n} \mu(d)\, n/d$$

$$= \sum_{n\le x} \sum_{d|n} \mu(d)\ n/d$$

$$= \sum_{\substack{d|n \\ n = qd \le x}} \mu(d)\, q \quad \text{(Since } q = n/d)$$

$$= \sum_{d\le x} \mu(d) \sum_{q\le x/d} q$$

$$= \sum_{d\le x} \mu(d) \{ ½ (x/d)^2 + 0(x/d)\}$$

$$(11) \qquad = ½\, x^2 \sum_{d\le x} \mu(d)/d^2 + O\,(x \sum_{d\le x} 1/d)$$

But

$$(12) \quad \sum_{d\le x} \mu(d)/d^2 = \sum_{d=1} \mu(d)/d^2 - \sum_{d>x} \mu(d)/d^2 = 6/\pi^2 + O\left(\frac{1}{x}\right)$$

Putting (12) in (11) we obtain

$$\sum_{n\le x} \varphi(n) = ½\, x^2 \{ 6/\pi^2 + O(1/x)\} + O\,(x \log x)$$

$$= (3/\pi^2)\, x^2 + O\,(x \log x).$$

This completes the proof .

2.8. COMPUTER PROGRAMMING FOR DETERMINIG PRIME FACTORIZATION OF N AND VALUE OF PHIN (Φ(N)), TAUN(τ(N)=d(N)) AND SIGMA (N) (σ(N))

```
C FOR ANY INTEGER N IN THE RANGE 2
C THROUGH 1, 073, 938, 400 THIS
C PROGRAM DETERMINES THE PRIME FACTORIZATION OF N
AND VALUES
C PHI (N), TAU (N), AND SIGMA (N)
DIMENSION NPR (3512), N FAC (10), NBR (10), N (20), INT (20), NQ
(20) DIMENSION NPHI (10), NSIG (10) LIMIT = 3512
10 FORMAT (1615)
```

```
15 READ (2.20) (NCI), 1 = 1, 20)
20 FORMAT (2011)
CALL DIVN (1, N, INT, NREM, M, K)
IF (M=20)  25, 23, 25
23 IF (N (20) = 2) 80, 70, 25
25       NRT = NROOT (INT, M. 20)
J = 0
DO 50 1 = 1, LIMIT.
NF = 0
IF (NPR (I) = NRT) 32, 32, 52
32 (ALL DIVN (NPR (1), INT, NQ, KINT, L. K)
IF (KINT) 36, 34, 36
34       DO 35. 11 = M, 20
35      INT (11) = NQ (11)
NF = NF +1
GO TO 32
36 IF (NF) 50, 50, 38,
38 J = J + 1
NFAC (J) = NPR (I)
NBR (J) = NF
NRT = NROOT (INT, 1, 20)
50 CONTINUE
51 IF (J) 70, 70, 60
60 WRITE (3, 61) (N (K), K = M, 20)
61 FORMAT (1 HO, 1011)
WRITE (3, 62) (NFAC (K)),
N B R (K) , K = 1,  J
62 FORMAT (1H +, 10 X, 3H = , 9 (15, 1H (j 12, 1H) ))
CALL PTS (NFAC, J, NBP, NPH 1, NTAU,  N SIG, NDP, NDS, O)
IF (L = 20), 66, 64, 66
64   IF (INT (20) = 1 ) 75, 75, 66
```

```
66 WRITE (3, 67) (INT (K), K = L, 20)
67 FORMAT (1H, 15 X, 1011)
K = 1
GO TO 73
70 WRITE (3, 71) (N (K), K = M, 20)
71 FORMAT ( 1 HO, 1011)
WRITE (3, 72)
72 FORMAT ( 1H +, 13 X, BHIS PRIME)
K = –1
73 DO 74 IA = 1, 10
74 NFAC (IA) = INT (IA + 10)
NBR (1) = 1
CALL PTS (NFAC, 1 , NBR, NPHI, NTAU, NSIG, NDP, NDS, K )
75 WRITE  (3, 76) (N PHI (K), K = NDP, 10)
76 FORMAT ( 1H, 5X, 6HTAV=, 15)
77 WRITE (3, 78) (NSIG (K), K = NDS, 10)
78 FORMAT (1H, 5X, 8HSIGMA = , 1011)
GO TO 15
80      CALL EXIT
END
```

RESULT

```
            405769 = 7 (4) 13 (2)
            PH 1 = 321048
            TAU = 15
            SIGMA = 46448640
            13123110 = 2 (1) 3 (1) 5 (1) 7 (2) 11 (1) 13 (1) 19 (1) 23 (1)
            PH 1 = 228 0969
            TAU = 256
            SIGMA = 46448640
            18061 IS PRIME
            PH 1 = 18060
```

```
TAU = 2
SIGMA = 18062
62742247 IS PRIME
PH1 = 62742246
TAU = 2
SIGMA = 62742248
62742255 = 3 (1) 5 (1) 599 (1) 6983 (1)
PH1 = 33401888
TAU = 16
SIGMA = 100569600
N = 62742267 = 3(2) 7(1) 995909(1)
= 35852688
TAU = 12
SIGMA = 103574640
```

EXERCISES

1. Evaluate : (i) φ (5186), *(ii)* φ (56800), *(iii)* φ(7208) , *(iv)* φ (640412), *(v)* φ (628), *(vi)* φ(1001).
2. If for $n > 1$, $\varphi(n) \mid n-1$, prove that n is square free.
3. Prove that $\mu(n)\,\mu(n+1)\,\mu(n+2)\,\mu(n+3) = 0$ for each positive integer n.
4. If $n = \prod_{i=1}^{r} p_i^{a_i}$ prove that

   (i) $\sum_{d|n} \mu(k)\, d(k) = (-1)^r$.

   (ii) $\sum_{d|n} \mu(d)\, \sigma(d) = (-1)^r\, p_1\, p_2 \ldots\, p_r$.

   (iii) $\sum_{d|n} |\mu(d)| = 2^r$.

4. Prove that $\sum_{d|n} 1/d = \sigma(n) / n$ for each integer $n \geq 1$.

6. Prove that

   (i) d(n) is odd if and only if n is a perfect square ,

(ii) $\sigma(n)$ is odd if and only if n is a perfect square or twice a perfect square .

7. Prove that $\sum_{k|n} (d(k))^3 = (\sum_{k|n} d(k))^2$ for each integer $n \geq 1$.

8. For any integer n, show that

   (i) $\sum_{k|n} \sigma(k) = \sum_{k|n} (n/k)\, d(k)$ and

   (ii)· $\sum_{k|n} (n/k)\sigma(k) = \sum_{k|n} k\, d(k)$.

9. For $k \geq 2$ show that

   (i) If $2^k - 1$ is prime then $2^{k-1}(2^k - 1)$ satisfies the equation $\sigma(n) = 2n$.

(ii) If $2^k - 3$ is prime, then $n = 2^{k-1}(2^k\ 3)$ satisfies the equa.ion $\sigma(n) = 2n + 2$.

10. Prove that:

    (i) If n is an even integer then $\varphi(2n) = 2\,\varphi(n)$

    (ii) $\varphi(3n) = 3\varphi(n)$ if and only if $3|n$.

    (iii) $\varphi(n) = \varphi(n+2)$ is satisfied by $n = 2\,(2p-1)$ whenever p and $2p-1$ are both odd prime.

11. Prove that if $a|b$ then $\varphi(a)|\varphi(b)$.

12. Prove that if n is a perfect number i. e. $\sigma(n) = 2n$ then $\sum_{d|n} 1/d = 2$.

13. Prove that $\sum_{d|n} \mu(d)\varphi(d) = \prod_{p|n} (2 - p)$.

14. Prove that $\prod_{d^2|n} \mu(d) = \mu^2(n)$.

15. Show that $\varphi(n) = n-1$ if and only if n is prime.

16. Find all values of n for which $\varphi(n) = 6$.

◆◆◆

CHAPTER

3

CONGRUENCES

3.1. DEFINITION AND BASIC PROPERTIES OF CONGRUENCES.

Gauss in his great work, the "*Disquisitiones Arithmaticae*" in 1801 introduced a remarkable notation and basic facts which simplifies many problems concerning divisibility of integers. In doing so he created a new branch of number theory called the theory of congruences which is discussed in this chapter.

DEFINITION : Given integers a, b, m with $m > 0$, we say that a is congruent to b modulo m, and we write

(1) $$a \equiv b \pmod{m},$$

if m divides the difference $a - b$. The number m is called the modulus of the congruence .

In other words, the congruence (1) is equivalent to the divisibility relation

$$m \mid a - b.$$

In particular, $a \equiv 0 \pmod{m} \Leftrightarrow m \mid a$.

Hence $a \equiv b \pmod{m} \Leftrightarrow a - b \equiv 0 \pmod{m}$.

We write $a \not\equiv b \pmod{m}$ if $m \nmid (a-b)$

and we say a and b are incongruent modulo m.

Congruences are of great practical importance in everyday life. For example today is 'Thursday' is a congruence property (modulo 7) of the number of days that have passed since some fixed date. Lecture lists or railway guides are table of congruences, in the lecture list the relevant moduli are 365, 7 and 24.

Example 1 : Suppose that a lecture is given on every alternate date (including sundays), and that the first lecture occurs on a Monday. When will a lecture first fall on a Tuesday ? If this lecture is $(x+1)^{th}$ then $2x \equiv 1 \pmod 7$ and we find by trial the least positive solution is $x = 4$.

Thus the fifth lecture will fall on a Tuesday and this will be the first that will do so.

Example 2 : Consider $x^2 \equiv 1 \pmod 8$.

We can find by trial the solutions which are $x = 1, 3, 5, 7 \pmod 8$.

The congruence is also called as modular arithmetic. So modular arithmetic or congruence is finite. Many interesting problems on divisibility, remainder which when divided by an integer can be carried out with the help of congruence.

BASIC PROPETIES OF CONGRUENCES

THEOREM 1 : Congruence '≡' is an equivalence relation. That is we have

(*a*) $a \equiv a \pmod m$ (Reflexivity).

(*b*) $a \equiv b \pmod m \Rightarrow b \equiv a \pmod m$ (Symmetry)

(*c*) $a \equiv b \pmod m$ and $b \equiv c \pmod m$

$\Rightarrow$ $a \equiv c \pmod m$ (Transitivity).

PROOF :

(*a*) follows since $m \mid a - a = 0$.

(*b*) follow from the fact that if $m \mid (a-b)$ then $m \mid (b-a)$

(*c*) If $m \mid (a-b)$ and $m \mid (b-c)$ then $m \mid (a-b) + (b-c) = a-c$ (divisibility property).

THEOREM 2 : If $a \equiv b \pmod m$ and $\alpha \equiv \beta \pmod m$, then we have

(*a*) $ax + \alpha y \equiv bx + \beta y \pmod m$ for all integers x and y

(*b*) $a\alpha \equiv b\beta \pmod m$

(*c*) $a^n \equiv b^n \pmod m$ for every positive integer n.

(*d*) $f(a) \equiv f(b) \pmod m$ for every polynomial f with integer co–efficients.

(*e*) If $d \mid m$, $d \mid a$ then $d \mid b$.

(*f*) $(a, m) = (b, m)$.

(*g*) If $a \equiv b \pmod m$ and $a \equiv b \pmod n$ where $(m, n) = 1$ then $a \equiv b$ (mod mn).

PROOF : (*a*) Since $m \mid (a–b)$ and $m \mid (\alpha–\beta)$ we have

$m \mid x\,(a–b) + y\,(\alpha – \beta) = (ax + \alpha y) – (bx + \beta y)$.

(*b*) $a\alpha – b\beta = \alpha\,(a–b) + b(\alpha–\beta) \equiv 0 \pmod{m}$ by (*a*) .

(*c*) Take $\alpha = a$ and $\beta = b$ in (*b*) and using induction on n we get (*c*).

(*d*) Using (*c*) and induction on the degree of f we get (*d*).

(*e*) Assume that $d > 0$. If $d \mid m$ then $a \equiv b \pmod{m}$.

But if $d \mid a$ then $a \equiv 0 \pmod{m}$ so $b \equiv 0 \pmod{m}$.

(*f*) Let $d_1 = (a, m)$ and $d_2 = (b, m)$. Then $d_1 \mid m$ and $d_1 \mid a$ so $d_1 \mid b$; hence $d_1 \mid d_2$. Similarly $d_2 \mid m$, $d_2 \mid m$ implied $d_2 \mid a$; hence $d_2 \mid d_1$.

Therefore $d_1 = d_2$ i.e. $(a, m) = (b, m)$.

(*g*) Since both m and n divide $a – b$ so does their product mn since $(m, n) = 1$. Hence proved..

Example 3 : An integer is divisible by 9 if and only if, the sum of its digits in its decimal representation is divisible by 9.

Suppose $n = a_o + 10\,a_1 + 10^2\,a_2 + \ldots.. + 10^k\,a_k$.

$10 \equiv 1 \pmod{9}\ 10^2 \equiv 1 \ldots\ldots.\ 10^k \equiv 1 \pmod{9}\ 10^k\,a_k$

$\equiv 1 \pmod{9}$

for $k = 0, 1, 2, \ldots k$

So

$$n \equiv a_o + a_1 + a_2 + \ldots.. + a_k \pmod{9}$$

Since $3 \mid 9$ an integer is divisible by 3 if and only if, the sum of its digit in its decimal representation is divisible by 3.

Example 4 : An integer is divisible by 4 if and only if the difference between the sum of digits in the odd places and the sum of digits in the even place is divisible by 11.

Suppose $n = a_o + 10a_1 + 10^2a_2 + 10^3a_3 + \ldots. + 10^ka_k$

$1 \equiv 1 \pmod{11} \Rightarrow a_o \equiv a_o \pmod{11}$

$10 \equiv -1 \pmod{11} \Rightarrow 10\,a_1 \equiv -a_1 \pmod{11}$

$10^2 \equiv 1 \pmod{11} \Rightarrow 10^2\,a_2 \equiv a_2 \pmod{11}$

$10^3 \equiv -1 \pmod{11} \Rightarrow 10^3\,a_3 \equiv -a_3 \pmod{11}$

$10^k \equiv +1 \pmod{11}$ if k is even

$10^k \equiv -1 \pmod{11}$ if k is odd .

Thus $a_o + 10\,a_1 + \ldots + 10^k\,a_k \equiv (a_o + a_2 + a_4 + \ldots..) - (a_1 + a_3 + \ldots)$ $\equiv$ (mod 11)

So n is divisible by 11 if

$$(a_o + a_2 + \ldots + a_{k\ 1}) - (a_1 + a_3 + \ldots + a_k) \equiv 0 \pmod{11}.$$

Example 5 : We have shown in the Chapter I that F_5 is composite and divisible by 641.

Here with the help of congruence we will show it.

$$F_5 = 2^{2^5} + 1 = 2^{32} + 1.$$

Now $2^2 \equiv 4,\ 2^4 \equiv 16,\ 2^8 \equiv 256,\ 2^{16} \equiv 65{,}536 \equiv 154$ (mod 641).

$$\text{So } 2^{32} \equiv (154)^2 = 23,716$$
$$\equiv 640 \pmod{641}$$
$$\equiv -1 \pmod{641}.$$

Therefore, $F_5 = 2^{32} + 1 \equiv 0$ (mod 641), so F_5 is composite.

Example 6 : What is the remainder when 5^{48} is divided by 12 ?

Now, $5^{48} = (5^2)^{24} = (25)^{24} \equiv (1)^{24}$ (mod 12)

$\equiv 1$ (mod 12).

Hence the remainder is 1.

Example 7 : Find the remainder when $1! + 2! + 3! + \ldots.. + 200!$ is divided by 4.

$$1! + 2! + 3! + \ldots\ldots + 200!$$
$$\equiv 1! + 2! + 3! \pmod 4 \text{ (since k! is divisible by 4)}$$
$$\equiv 1 + 2 + 6 \equiv 1 \pmod 4 \text{ for } k \geq 4$$

Hence the remainder is 1 .

Example 8 : Determine the remainder when 2^{36} is divided by 11.

Solution : $2^{11} \equiv 2$ (mod 11)

$\Rightarrow$ $(2^{11})^3 \equiv 2^3$(mod 11)

$$2^{36} = 2^{33}.2^3$$
$$\equiv 2^3.2^3 \pmod{11}$$
$$\equiv 64 \pmod{11} \equiv 9 \pmod{11}.$$

Hence the remainder is 9.

Exercise

1. Find the last two digits of the number 9^{9^9} .

Example 9 : Find the remainder when 2^{14} is divided by 17.

$$2^4 \equiv -1 \pmod{17} \Rightarrow 2^8 \equiv 1 \pmod{17}. \text{ Again } 2^6$$

$$= 64 \equiv 13 \pmod{17}$$

$$2^{14} = 2^8 . 2^6 \equiv 64 \pmod{17} \equiv 13 \pmod{17}.$$

Congruence is like equality '=' in arithmetic. All the properties of addition, subtraction, multiplication are true for congruence as shown in theorem 2 . But the division is not always true for congruence. For example

$$18 \equiv 8 \pmod{10} \Rightarrow 9 \not\equiv 4 \pmod{10} \text{ since } 10 \nmid (9-4) = 5$$

Example 10 : (*a*) Find the remainder when 2^{50} and 41^{65} are divided by 7.

(*b*) What is the remainder when the sum $1^5 + 2^5 + 3^5 \ldots\ldots + 99^5 + 100^5$ is divisible by 4?

Solution :

(*a*) $2^3 \equiv 1 \pmod 7,\ 2^2 \equiv 4 \pmod 7$

These two imply $2^5 \equiv 4 \pmod 7,\ 2^{10} \equiv 4^2 \pmod 7 \equiv 2 \pmod 7$

Hence $2^{50} \equiv 2^5 \pmod 7 \equiv 4 \pmod 7$.

Hence the remainder is 4.

Again $41 \equiv (-1) \pmod 7 \Rightarrow (41)^{65} \equiv (-1)^{65} \pmod 7$

$$\equiv -1 \pmod 7$$

$$\equiv 6 \pmod 7.$$

The remainder is 6.

(*b*)

$$1 \equiv 1 \pmod 4 \Rightarrow 1^5 \equiv 1^5 \pmod 4 \quad 2^5 = 0 \pmod 4$$

$$3 \equiv -1 \pmod 4 \Rightarrow 3^5 \equiv (-1)^5 \pmod 4$$

$$4^5 \equiv 0 \pmod 4 \ldots\ldots 100 \equiv 0 \pmod 4).$$

All the even powers are divisible by 4. The odd terms are congruent to either 1 or –1 modulo 4 . Hence

$$1^5 + 2^5 + 3^5 + \ldots. + 99^5 + 100^5 \equiv 1^5 + (-1)^5 \pmod 4$$

$$\equiv (1-1) \pmod 4$$

$$\equiv 0 \pmod 4).$$

Example 11 : $2222^{5555} + 5555^{2222}$ is divisible by 7.

We know

$$2222 \equiv 3 \pmod 7$$

and $5555 \equiv 4 \pmod 7$.

$$5555 \equiv 5 \pmod 6$$
$$2222 \equiv 2 \pmod 6).$$

Thus $2222^{5555} + 5555^{2222} \equiv 3^5 + 4^2$

$$\equiv 12 + 2 \pmod 7)$$
$$\equiv 0 \pmod 7).$$

Example 12 : Using congruence we can also show that

(*i*) $13 | 3^{n+2} + 4^{2n+1}$ (*ii*) $43 | 6^{n+2} + 7^{2n+1}$

Example13 : Show that $41 | 2^{20} - 1$.

$$2^5 \equiv -9 \pmod{41} \Rightarrow (2^5)^4 \equiv (-9)^4 \pmod{41}$$

$\Rightarrow$ $2^{20} \equiv 81.81 \pmod{41}$ But $81 \equiv -1 \pmod{41}$.

Hence $2^{20} - 1 \equiv 81.81 - 1 \equiv 1 - 1 \equiv 0 \pmod{41}$.

In the following theorem we will show when this cancellation law is true for congruence. We show that a common factor can be cancelled if the modulus is also divisible by this factor.

THEOREM 3 : If $c > 0$ then $a \equiv b \pmod m$ if and only if

$$ac \equiv bc \pmod{mc}.$$

PROOF : We have $m | (b-a)$ if and only if $mc|c(b-a)$, i. e. $ac \equiv bc$ (mod mc)

THEOREM 4 : (Cancellation law) If $ac \equiv bc$ (mod m) and if $d = (m, c)$, then

$$a \equiv b \pmod{m/d}$$

If $d = 1$ we get

$$a \equiv b \pmod m.$$

That is, we can cancel the common factor which is relatively prime to the modulus .

PROOF : Since $ac \equiv bc \pmod m$ we have

$$m | c(a-b) \Rightarrow m/d | c/d(a-b)$$

But $(m, c) = d \Rightarrow (m/d, c/d) = 1$. Hence by Euclid's lemma $m/d | (a-b)$ i.e. $a \equiv b \pmod{m/d}$. Hence proved .

3.2. RESIDUE CLASSES AND COMPLETE RESIDUE SYSTEM

DEFINITION : Consider a fixed modulus $m > 0$. We denote by $\hat{a}$ the set of integers x such that $x \equiv a \pmod m$

i.e. $\hat{a} = \{ x: x \text{ is an integer and } x \equiv a \pmod{m}\}$.

We call $\hat{a}$ the residue class a modulo m. Thus $\hat{a}$ consists of all integers of the form $a + mk$, $k = 0, \pm1, \pm2 \ldots$ The following property is satisfied by the residue classes.

Example : $\hat{0}, \hat{1}, \hat{2}, \ldots \hat{9}$ are the residue classes modulo 10.

THEOREM 5 : For a given modulus m we have

(*a*) $\hat{a} = \hat{b}$, if and only if, $a \equiv b \pmod{m}$

(*b*) Two integer x and y are in the same residue class if and only if $x \equiv y \pmod{m}$

(c) The m residue classes $\hat{1}, \hat{2} \ldots \hat{m}$ are disjoint and their union is the set of all integers.

PROOF :

(*a*) $\hat{a} = \hat{b} \Leftrightarrow x \equiv a \pmod{m} = b \pmod{m} \Leftrightarrow a \equiv b \pmod{m}$

(*b*) $x \equiv y \pmod{m} \Leftrightarrow \hat{x} = \hat{y}$ by (*a*). Hence two integers x and y are in the same residue classes.

(*c*) To prove (*c*) we note that the numbers 0, 1, 2, … m–1 are incongruent module m. Hence the residue classes $\hat{0}, \hat{1} \ldots \widehat{m\text{-}1}$ are disjoint But every integer x must be in exactly one of these classes because $x = qm + r$ where $0 \le r < m$, so $x \equiv r \pmod{m}$ and hence $x \in \hat{r}$. Since $\hat{0} = \hat{m}$this proves (*c*).

DEFINITION : A set of m representative one from each of the residue classes $\hat{1}, \hat{2} \ldots \hat{m}$ is called a COMPLETE RESIDUE SYSTEM modulo m. We write in short as CRS modulo m.

Example : (*i*) {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10} is a CRS (mod 11).

(*ii*) $\{ 1, m+2, 2m+3, 3m+4, \ldots\ldots m^2\}$

{ 0, 1, 2, …….m–1 }, { 1, 2, 3, …. m } are the set m integers, incogruent module m are CRS modulo m.

3.3. REDUCED RESIDUE SYSTEM AND FERMAT'S THEOREM

DEFINITION: The set of φ(m) integers incongruent modulo m, each of which is relatively prime to m is a REDUCED RESIDUE SYSTEM modulo m. We write in short RRS modulo m .

Example 1 : If m = 8 then

$\{ 1, 3, 5, 7 \}$ form the RRS modulo 8.

Also $\{ 9, 11, 13, 15 \}$ form RRS modulo 8.

THEOREM 6 : Assume (k, m) = 1 .If $\{ a_1\ a_2 \dots a_m\}$ is complete residue system modulo m, so is $\{ k\,a_1, k\,a_2, \dots, k\,a_m\}$.

PROOF : We will show that the elements of the set $\{ka_1, ka_2 \dots ka_m\}$ are distinct and mutually incongruent modulo m. Suppose they are not, then $ka_i = ka_j$ (mod m) since (k,m)=1 by cancellation law $a_i \equiv a_j$ (mod m) which is not true by the hypothesis that $\{ a_1, a_2 \dots a_m\}$ is a CRS mod m . Therefore two elements in the set $\{ka_1, ka_2 \dots ka_m\}$ are mutually incongruent modulo m. Since there are m elements in this set it forms a CRS modulo m.

THEOREM 7 : Assume (k, m) = 1. If $\{a_1, a_2 \dots a_{\varphi(m)}\}$ is a reduced residue system modulo m, then $\{ka_1, ka_2 \dots ka_{\varphi(m)}\}$ is also a reduced residue system modulo m..

PROOF : Since $\{ a_1, a_2 \dots a_{\varphi(m)}\}$ is a RRS modulo m by definition (a_i m) = 1 for i = 1,2, ...φ(n). It is given that (k, m) = 1. These two imply (ka_i, m) = 1. That is each ka_i, i = 1, 2... is relatively prime to m.

So there are φ(m) in number.

Now $ka_i \equiv ka_j$ (mod m), (k,m) = 1

$\Rightarrow$ $a_i \equiv a_j$ (mod m) which is false since $\{a_1, a_2 \dots a_{\varphi(m)}\}$ runs through a RRS modulo m . Hence $\{ka_1, ka_2, \dots ka_{\varphi(m)}\}$ also forms a RRS modulo m.

THEOREM 8 : (Fermat's Little Theorem)

For any integer a and any prime p we have

$$a^p \equiv a \pmod{p}.$$

This will follow from the generalised version called Euler Fermat theorem.

THEOREM 9 : (Euler Fermat-theorem)

Assume (a, m) = 1. Then we have $a^{\varphi(m)} \equiv 1 \pmod{m}$.

PROOF : Let $\{ b_1, b_2 \dots b_{\varphi(m)}\}$ be a reduced residue system modulo m Then $\{ab_1, ab_2 \dots ab_{\varphi(m)}\}$ is also a reduced residue system modulo m. Hence the product of all the integer in the first set is congruent to the product of those in the second set. Therefore

$$b_1\, b_2 \,....\, b_{\varphi(m)} \equiv ab_1, ab_2 \,...\, ab_{\varphi(m)}$$
$$\equiv a^{\varphi(m)}\, b_1\, b_2 \,.....\, b_{\varphi(m)} \text{ (mod m).}$$

Since each b_i is relatively prime to m $(b_1, m) = 1$ for i = 1, 2... $\varphi(m)$ canceling each b_i we get $a^{\varphi(m)} \equiv 1$ (mod m).

Corollary 1 : If a prime p does not divide a then $a^{p-1} \equiv 1$ (mod m).

Taking m = p, $\varphi(m) = \varphi(p) = p-1$, and (a, p) = 1, hence the result follows from theorem 9.

PROOF OF THEOREM 8 : If a is any integer and p is any prime then two cases arise.

Either p | a or p+a. In the case when p | a

we get p | a^p hence p | a^p–a, i e $a^p - a \equiv 0$ (mod p).

Or $a^p \equiv a$ (mod p).

In the second case when p $\nmid$ a i.e. (a, p) = 1 then by the corollary

we get $a^{p-1} \equiv 1$ (mod p).

Multiplying by a both sides since (a, p) = 1, we get $a^p \equiv a$ (mod p)

Thus theorem 8 is proved..

SECOND PROOF OF THEOREM 8 : The proof is by induction on a. If a = 1 then $1^p \equiv 1$ (mod p) is true as is the case a = 0 . Assume the result hold for a, we will show it is true for a+1. that is to show that $(a+1)^p = (a+1)$ mod p when $a^p \equiv a$ (mod p).

Now, by binomial theorem

$$(a+1)^p = a^p + \binom{p}{1} a^{p-1} + \ldots + \binom{p}{1} a^{p-k} \ldots + \binom{p}{p-1} a + 1.$$

By assumption $a^p \equiv a$ mod p.

We have to show that the binomial coefficients $\binom{p}{k}$ given by

$$\binom{p}{k} = p!/k!(p-1)! \text{ is divisible by p for } k = 1.\ 2\ ....\ p-1.$$

That is to show that

$$\binom{p}{k} \equiv 0 \text{ (mod p).}$$

Now $k! \binom{p}{k} = p!/(p-k)! = p\,(p-1)\ldots(p-k+1) \equiv 0 \pmod{p}$,

which implies either $p \mid k!$ or $p \mid \binom{p}{k}$. But $p \mid k!$ is impossible since $1 \leq k \leq p-1$.

Hence $p \mid \binom{p}{k}$ i. e. $\binom{p}{k} \equiv 0 \pmod{p}$.

Hence we have

$$(a+1)^p \equiv a^p + 1 \equiv (a+1) \pmod{p}.$$

Hence the theorem follows when $a \geq 0$. If a is negative integer, there is no problem since $a = r \pmod{p}$ for some r, where $0 \leq r \leq p-1$.

We get $a^p \equiv r^p = r \equiv a \pmod{p}$.

THEOREM 10 : If p is prime, then $(x+y+\ldots+w)^p \equiv x^p + y^p + \ldots + w_p \pmod{p}$.

PROOF : $(x+y)^p = x^p + \binom{p}{1} x^{p-1} y + \ldots + \binom{p}{p-1} xy^{p-1} + y^p \equiv x^p + y^p \pmod{p}$,

since $\binom{p}{k}$, $k = 1, 2, 3\ldots p-1$ is divisible by p,

the general result follows by the repetition of the argument .

Corollary 2 : $a^p \equiv a \pmod{p}$ (**Little Fermat's theorem**).

By taking $x = y = z = \ldots = w = 1$ in theorem 11 if there are a number we get

$$(1+1+\ldots+a)^p \equiv 1^p + 1^p \ldots \text{ to a} \pmod{p}$$

$$\Rightarrow \qquad a^p \equiv a \pmod{p}.$$

THEOREM 11 : If $\alpha > 0$ and $m \equiv 1 \pmod{p^\alpha}$

then $m^p \equiv 1 \pmod{p^{\alpha+1}}$

PROOF : $m \equiv 1 \pmod{p^\alpha} \Rightarrow$

$m = 1 + kp^\alpha$, k is an integer, and $\alpha p \geq \alpha + 1$.

Hence

$$m^p = (1 + kp^\alpha)^p = 1 + l\,p^{\alpha+1} \text{ where } l \text{ is an integer.}$$

Corollary 3 : $a^{p(p-1)} \equiv 1 \pmod{p^2}$,

$$a^{p^2(p-1)} \equiv 1 \pmod{p^2},\ a^{p^{(\alpha-1)}(p-1)} \equiv 1 \pmod{p^2}$$

Taking $m = a^{p-1}$ in theorem 11 we have $a^{p-1} \equiv 1 \pmod p$ then we get $a^{p(p-1)} \equiv 1 \pmod{p^2}$. Again taking $m = a^{p(p-1)}$ and $\alpha = 2$ we get

$a^{p^2(p-1)} \equiv 1 \pmod{p^2}$ and so on. Hence proved .

Example 1 : $11 \mid 5^{38} - 4$.

By Fermat's theorem $5^{10} \equiv 1 \pmod{11}$ since $(5,11) = 1$.

Now $5^{38} = 5^{10\times3+8} = (5^{10})^3\,(5^2)^4$

$$\equiv 1.\,(25)^4 \equiv 1.3^4 \equiv 81 \equiv 4 \pmod{11}.$$

Since $25 \equiv 3 \pmod{11}$

Hence $11 \mid 5^{38}-4$.

Example 2 : Find the unit digit of 3^{100}, by Femat's theorem.

Fermat's theorem gives $3^4 \equiv 1 \pmod 5$, $3^4 \equiv 1 \pmod 2$.

These two $\Rightarrow 3^4 \equiv 1 \pmod{10}$

$\Rightarrow 3^{100} \equiv 1 \pmod{10}$

$\Rightarrow$ 1 is the unit digit in 3^{100}.

Example 3 : Find the last two digits of 3^{100}.

Since $(3, 100) = 1$, $3^{\varphi(100)} \equiv 1 \pmod{100}$

But $\varphi(100) = 40$. We have

$$\begin{aligned}
3^{40} &\equiv 1 \pmod{100} \\
3^{100} &= 3^{40.\,2+20} = (3^{40})^2\,3^{20} \\
&\equiv 3^{20} \pmod{100} \\
&\equiv (3^5)^4 \pmod{100} \\
&\equiv (43)^4 \pmod{100} \equiv (1849)^2 \pmod{100}\,. \\
&\equiv (49)^2 \pmod{100} \\
&\equiv 01 \pmod{100}\,.
\end{aligned}$$

Example 4 : Determine the last two digits of 1999^{1999}.

We need congruence $1999^{1999} \pmod{100}$.

$$\begin{aligned}
\varphi(100) &= \varphi(25)\,\varphi(4) = 40. \\
1999 &= 40k+39,\ 1999 \equiv 99 \pmod{100} \\
1999^4 &\equiv 1 \pmod{100} \text{ (By Euler's–Fermat's theorem)} \\
1999^{40k} &\equiv 1 \pmod{100}.
\end{aligned}$$

Hence $1999^{40k+39} \equiv (99)^{39}$ (mod 100).

(By Euler–Fermat theorem) $99^{40} \equiv 1$ (mod 100).

To find $(99)^{39}$ (mod 100)

$$99 \equiv -1 \pmod{100}$$

Hence $\quad (99)^{39} \equiv (-1)^{39} \pmod{100}$

$$\equiv -1 \ (\text{mod } 100\ ) \equiv 99 \pmod{100}$$

That is, the last two digits are 99.

Example 5 : Find the last two digits of $N = 123^{456}$.

Solution : $\quad 123 \equiv 23$ (mod 100). We know $a^{2^{n+1}} = (a^{2^n})^2$

We have $\quad 456 = 256 + 128 + 64 + 8$.

$$123 \equiv 23 \pmod{100}$$

$$(123)^2 \equiv 23^2 = 529 \equiv 29 \pmod{100}$$

$$(123)^4 \equiv 29^2 \equiv 41 \pmod{100}$$

$$(123)^8 \equiv 41^2 \equiv 81 \pmod{100}$$

$$(123)^{16} \equiv 81^2 \equiv 61 \pmod{100}$$

$$(123)^{32} \equiv 61^2 \equiv 21 \pmod{100}$$

$$(123)^{64} \equiv 21^2 \equiv 41 \pmod{100}$$

$$(123)^{128} \equiv 41^2 \equiv 81 \pmod{100}$$

$$(123)^{256} \equiv 81^2 \equiv 61 \pmod{100}.$$

Therefore $123^{456} \equiv 123^{256+128+64+8} = 61.\ 81.41.\ 81 = 16409061 \equiv 61$ (mod 100)

Exercise : Determine the remainder when 2^{720} is divided by 225.

We can use for Euler–Fermat theorem to find the inverse of a mod m, we define the inverse of a mod m as:

Let m be a fixed integer. The arithmetic inverse of a given integer a such that (a, m) = 1 is an integer a^* such that $aa^* \equiv 1$ (mod m).

The integer a^* is called as inverse of a mod m.

Example 2 : If m = 10, the inverse of 3 is 7 since 3.7 = 21 – 1 (mod 10).

Since $a^{\varphi(m)} \equiv a.a^{\varphi(m)-1} = 1$ (mod m), $a^{\varphi(m)\ 1}$ is an inverse of a modulo m.

Example 3 : If m = 35, φ (35) =φ(7), φ(5) = 6.4= 24. Hence $2^{24} \equiv 1$ (mod 38).

The inverse of 2 mod 35 must be 2^{23} (mod 35).

To calculate 2^{23} (mod 35)

$$2^6 \equiv 1 \pmod 7 \Rightarrow 2^{12} \equiv 1 \pmod 7$$

$$2^4 \equiv 1 \pmod 5 \Rightarrow 2^{12} \equiv 1 \pmod 5$$

These two imply $2^{12} = 1$ (mod 35).

$$2^{23} = 2^{12+11} \pmod{35} \equiv 2^{11} \pmod{35} \equiv 18 \pmod{35}$$

Hence the inverse of 2 mod 35 is 18.

Exercise : Determine the inverse of 3 modulo 40.

Example 4 : Express a^{37} as a product of power of a where the exponents are powers of 2.

We write 37 in base 2 as 37 = 32+4+1 =

$$= 1.2^5 + 0.2^4 + 0.2^3 + 1.2^2 + 0.2 + 1.2^0$$

$$= 100101 \text{ in base 2.}$$

There fore $a^{37} = a^{32}.\ a^4.\ a^1$.

We describe an algorithm in which the base 2 expansion of k in a^k is implicit. The idea is to traverse the digits of k (in base 2) from right to left. Every time we encounter a 0, divide k by 2 and square n, but do not add the term to the result. If k is odd, we multiply the result by the current power of a and subtract 1 from 4.

ALGORITHM (Exponential modulo m) Given integers, k and m this algorithm compute a^k mod m for k > 0.

1. [Initialize] set result = 1
2. [Check if done]. If k = 0, return result and terminate .
3. [k is odd]. If k mod 2 = 1, then let result = (result a) mod m; k = k - 1, and go to step 2.

4 [k is even]. Let a = a^2 mod m, k = k/2 and go to step 2.

Example 5 : Let k = 37, the values in the computation are as follows.

TABLE

Iteration	a	Result	k base 10	k base 2
0	a	1	37	100101
1	a	a	36	10000
2	a^2	a	18	10010
3	a^4	a	9	1001
4	a^4	a^5	8	1000
5	a^8	a^5	4	100
6	a^{16}	a^5	2	10
7	a^{32}	a^5	1	1
8	a^{32}	a^{37}	0	0

APPLICATION OF THE METHOD OF COMPUTING a^k MODm

We will apply this method to determine if a number n is a prime power. If $n = p^k$, Fermat's theorem implies that $a^n \equiv a \pmod{p}$, then $d = (a^n - a, n)$ is divisible by p. If d is not prime then it is a power of prime we now check if d is a power of prime, if it is, then we can repeatedly divide n by this prime to see if n is a prime number. In the computation of d, it is sufficient to compute $a^n - a$ mod n.

Suppose n =28561 and a = 2, $a^n - a \bmod n = 2^n - 2 \pmod{28561}$

=4810, (4810, n) = 13. Hence n could be a power of 13.

Repeatedly dividing n by 13 we have $n = 13^4$.

Example 6 : Find the last two digits in the decimal representation of 3^{256}.

Solution : To find $3^{256} \pmod{100}$.

Now $\varphi(100) = 40$,

$$3^{40} \equiv 1 \pmod{100} \quad 256 = 6 \times 40 + 16$$

$$3^{256} \equiv 3^{6 \times 40 + 16} \equiv (3^{40})^6\, 3^{16} \equiv 3^{16} \pmod{100}$$

$$\text{Now } 3^{16} \equiv (81)^4 \equiv (-19)^4 \equiv (361)^2$$

$$\equiv (61)^2 \equiv 21 \pmod{100}.$$

Now the question is whether the converse of Fermat's theorem is true or not . That is $a^n \equiv 1 \pmod{n}$ implies n is a prime. This is not true.

For example if $n = 117, a = 2$, we write

$$2^{117} = 2^{7.16+5} = (2^7)^{16} 2^5$$

$$\text{and } 2^7 = 128 \equiv 11 \pmod{117},$$

we have $2^{117} \equiv 11^{16} 2^5 \equiv (121)^8 2^5 \equiv 4^8 2^5 \equiv 2^{21} \pmod{117}$

But $2^{21} = (2^7)^3$ which gives

$$2^{21} \equiv 11^3 \equiv 121.11 \equiv 4.11 \equiv 44 \pmod{117}$$

We finally get

$$2^{117} \equiv 44 \not\equiv 2 \pmod{117}$$

so that 117 is composite and 117 = 13.9. The converse of Fermat's theorem is not true. That is $a^{n-1} \equiv 1 \pmod n$ for some integer a then n need not be a prime. We will illustrate it by an example. For this we require the following lemma .

LEMMA : If p and q are distinct primes such that $a^p \equiv a \pmod q$ and $a^q \equiv a \pmod p$, then $a^{pq} \equiv a \pmod{pq}$.

PROOF : It is known from Fermat's theorem $(a^q)^p \equiv a^q \pmod p$ while $a^q \equiv a \pmod p$ by hypothesis . Combining these two we get $a^{pq} \equiv a \pmod p$ or $p \mid a^{pq} - a$.

Similarly $q \mid a^{pq} - a$. This yields $pq \mid a^{pq} - a$ which we write $a^{pq} \equiv a$ (mod pq).

Example : We will now show that $2^{340} \equiv 1 \pmod{341}$ where 341 = 11.31, a composite number.

$$2^{10} = 1024 = 31.33 + 1. \text{ Thus } 2^{11}$$

$$= 2.2^{10} \equiv 2.1 \equiv 2 \pmod{31}$$

and $2^{31} = 2\,(2^{10})^3 \equiv 2.1^3 \equiv 2 \pmod{11}$

By the lemma, $2^{11}\, 2^{31} \equiv 2 \pmod{11.31}$

or $2^{341} \equiv 2 \pmod{341}$.

After canceling common factor 2 we get $2^{340} \equiv 1 \pmod{341}$

So that the converse of Fermat's theorem fails .

3.4. LINEAR CONGRUENCES

Polynomial congruences are analogous to the polynomial equation in algebra. We consider the polynomial with integer coefficients so that the values of these polynomial will be integers when x is an integer. Consider $f(x) \equiv 0 \pmod m$, x, m integers . Then x is called as a solution of the congruence. By the properties of congruence if $x \equiv y \pmod m$ then $f(x) \equiv f(y) \pmod m$. So every congruence having one solution has infinitely many solutions, we will consider solutions which are distinct and incongruent modulo m. So the number of solution are contained in the set {1, 2, ... m} or any other complete residue system modulo m . So it is expected as in case of polynomial in algebra that every polynomial congruence modulo m has at most m solution, But it is not always true . There are some polynomial congruence which has no solution. A polynomial congruence of degree one is called as linear congruence.

Example : The linear congruence $2x \equiv 3 \pmod 4$ and $2x \equiv 1 \pmod 2$ have no solution, since $2x-3$ is odd for every x and can not be divisible by 4. Similarly in other case $2x-1$ is odd and can not be divisible by 2.

Example : A quadratic congruence $x^2 \equiv 1 \pmod 8$ has exactly four solutions i.e. has more solution than the degree of the polynomial congruence i.e. 2. The solutions are given by $x \equiv 1, 3, 5, 7 \pmod 8$.

We will consider now linear congruences, we will see when it has exactly one solution, what is the condition that it must have solution and what is total number of solutions and how to find these solutions.

THEOREM 12 : Assume $(a, m) = 1$. Then the linear congruence

(2) $ax \equiv b \pmod m$ has exactly one solution.

PROOF : As the distinct incongruent solution modulo m are among the CRS mod m, we test the numbers 1, 2,...m . Now form the product with a i.e. a, 2a . . . . . ma. Since $(a, m) = 1$ by theorem 6 these numbers also constitute a CRS modulo m. That is there is one x satisfying (2) i.e. $ax \equiv b \pmod m$.

Hence proved .

This theorem tells us the existence of exactly one solution but it does not say anything about how to determine this solution. If $(a, m) = 1$ then the unique solution of the congruence $ax \equiv 1 \pmod m$ is the reciprocal of a mod m. About the reciprocal we have already studied in the section 3.3.

To see the necessary and sufficient condition for the congruence (2) has solutions.

THEOREM 13 : Assume $(a, m) = d$. Then the linear congruence (2) has solution if, and only if, $d \mid b$.

PROOF : If a solution of linear congruence (2) exists then $(a, m) = d$. implies $d \mid m$ and $d \mid a$. Hence $d \mid ax-my = b$ i.e. $d \mid b$.

Conversely dividing the congruence (2) $ax \equiv b \pmod m$ by d ,

we get

(3) $a/d \; x \equiv b/d \pmod{m/d}$ and

$(a, m) = d \Rightarrow (a/d, m/d) = 1.$

Therefore by theorem 12 congruence (3) has exactly one solution which is also a solution of (2).

Example : $2x \equiv 3 \pmod 4$ has no solution because $(2,4) = 2$ and $2 \nmid 3$.

THEOREM 14 : Assume $(a, m) = d$ and suppose that $d \mid b$. Then the linear congruence (2) has exactly d solutions modulo m. These are given by

(4) $t, t + m/d, t + 2\, m/d, \ldots\, t + (d-1)\, m/d,$

where t is the solution unique modulo m/d, of the linear congruence (2).

PROOF : A solution of (3) is a solution of (2) since if x satisfies (3)

i.e. $a/d\ x \equiv b/d \pmod{m/d}$ then multiplying through out by d,

we get $ax \equiv b \pmod m$ i.e. (2) is satisfied conversely if (2) is satisfied then (3) is satisfied. That is if x satisfies.

$$ax \equiv b \pmod m$$

and since $(a, m) = d$, dividing by d we get $a/d\ x \equiv b/d \pmod{m/d}$ i.e. (3) is satisfied .

Now the d number of solution given in the set (4) are of the solution of (3) hence of (2). No two of these are congruent modulo m since the relation.

$$t + r\, m/d \equiv t + s\, m/d \pmod m,$$

with $0 \le r < d, 0 \le s < d.$ imply

$$r\, m/d \equiv s\, m/d \pmod m$$

and hence,

$$r \equiv s \pmod d \text{ i.e. } d \mid r-s \text{ but } |r-s| < d \text{ so } r = s.$$

Next to show that the linear congruence (2) has no solution except those listed in (4). If y is a solution of (2) then $ay \equiv at \pmod m$ so $y \equiv t \pmod{m/d}$ by cancellation law. That is $y = t + km/d$ for some integer k. But $k \equiv r \pmod d$ for some r satisfying $0 \le r < d$. Therefore

$$k\, m/d \equiv r\, m/d \pmod m.$$

So $$y \equiv t + rm/d \pmod m.$$

Therefore y is congruent modulo m to one of the number in the set (4). This completes the proof.

Now we will establish a theorem for determining all solution of the linear congruence. For this we use Euler Fermat theorem.

THEOREM 15 : If $(a, m) = 1$ the solution (unique mod m) of the linear congruence

(5) $ax \equiv b \pmod{m}$ is given by

(6) $x \equiv ba^{\varphi(m)-1} \pmod{m}$.

PROOF : The number x given by (6) satisfies (5) because

$$\begin{aligned} ax &\equiv a.ba^{\varphi(m)-1} \pmod{m} \\ &\equiv ba^{\varphi(m)} \pmod{m} \\ &\equiv b \pmod{m} \end{aligned}$$

by Euler– Fermat theorem. Since (a, m) = 1 the solution is unique mod m. If (a, m) =d we get d number of solution t = k m/d, k = 0, 1,2, 3... (d – 1) and t is given by (6). We illustrate these by examples.

Example 1 : Solve the congruence $3x \equiv 5 \pmod{20}$.

Solution : Since the g c d (3, 20) = 1. there is unique solution mod 20. By (5) the solution is

$$x \equiv 5.3^{\varphi(20)-1} \equiv 5.3^7 \pmod{20}$$

But $3^4 \equiv 1 \pmod{20}$

and $3^7 \equiv 3^3 \pmod{20}$

so $x \equiv 5.3^3 \pmod{20}$

$\equiv 5 \times 27 \pmod{20}$

$\equiv 15 \pmod{20}$.

The solution is 15 (mod 20).

Example 2 : Solve $27x \equiv 45 \pmod{180}$.

Solution : Since $d = (27, 180) = 9$, and $9 \mid 45$,

so the congruence has 9 solution modulo 180.

Dividing by 9 we get

$3x \equiv 5 \pmod{20}$, which has solution 15 .

The solutions are given by

$$x = 15 + 20k, k = 0, 1, 2, 3, 4, 5, 6, 7, 8$$

They are

$x \equiv 15, 35, 55, 75, 95, 115, 135, 155, 175, \pmod{180}$.

Example 3 : Solve the linear congruence $9x \equiv 12 \pmod{15}$.

Solution : $d = (9, 15) = 3$

The linear congruence has solution since $3 \mid 12$ and it has 3 solutions

Dividing the congruence by 3 we get

$$3x \equiv 4 \pmod{5}.$$

The solution of this linear congruence is

$$x \equiv 4.3^{\varphi(5)-1} \pmod{5} = 4.3^3 \pmod{5} \equiv 3 \pmod{5}.$$

Hence the solution of the given linear congruence are

$$3 + 5k,\ k = 0, 1, 2. \text{ i.e. are } 3, 8, 13 \pmod{15}.$$

3.5. POLYNOMIAL CONGRUENCES MODULO p, LAGRANGE'S THEOREM :

The fundamental theorem of algebra states that every polynomial f(x) of degree $n \geq 1$ the equation $f(x) = 0$ has n solution. There is no direct analog of this theorem for polynomial congruences. We have already shown in examples that some linear congruences have no solutions, some have exactly one solution and some have more than one. However, for congruences modulo a prime p we have the following theorem which guarantees p number of solutions of the congruence $f(x) \equiv 0 \pmod{p}$.

THEOREM 16 : (Lagrange)

Let $f(x) = c_0 + c_1 x + \ldots + c_n x^n$ be a polynomial of degree n with integer coefficients such that $c_n \neq 0 \pmod{p}$, p is a prime. Then the polynomial congruence

(7) $$f(x) \equiv 0 \pmod{p}$$

has at most n solutions.

PROOF : We use the induction on n, the degree of f. When n = 1 the congruence is linear $c_1 x + c_0 \equiv 0 \pmod{p}$.

Since $c_1 \neq 0 \pmod{p}$ i.e. $(c_1, p) = 1$, it has exactly one solution. Assume that the theorem is true for polynomials of degree n – 1. Suppose that the equation (7) has n + 1 incongruent solution modulo p, say $x_0, x_1 \ldots x_n$. ie $f(x_k) \equiv 0 \pmod{p}$ for each $k = 0, 1, \ldots. n$.

Now
$$\begin{aligned} f(x) - f(x_0) &= c_0 + c_1 x + \ldots + c_n x^n \\ &\quad -(c_0 + c_1 x_0 + \ldots + c_n x_0^n) \\ &= c_1 (x - x_0) + \ldots + c_n (x^n - x_0^n) \\ &= (x - x_0)\, g(x), \end{aligned}$$

where g (x) is a polynomial of degree n–1 with integer coefficients. By our assumption g (x) has (n–1) solutions. Thus

$$f(x_k) - f(x_0) = (x_k - x_0)\, f(x_k) \equiv 0 \pmod{p}$$

since
$$f(x_k) \equiv f(x_0) \equiv 0 \pmod{p}.$$

But $\quad x_k - x_o \equiv 0 \pmod p$ if $k \neq 0$.

So we must have $g(x_k) \equiv 0 \pmod p$ for each $k \neq 0$.

That is $g(x) \equiv 0 \pmod p$ has n incongruent solutions modulo p contradicting our induction hypothesis. This completes the proof.

Remark : The theorem fails if we replace prime modulo p by a composite modulo m. For example $x^2 \equiv 1 \pmod 8$ has four solution 1, 3, 5, 7, instead of two solutions.

3.6. APPLICATION OF LAGRANGE'S THEOREM :

We will describe here some applications of Lagrange's Theorem.

THEOREM 17 : If $f(x) = c_o + c_1 x + \ldots + c_n x^n$ is a polynomial of degree n with integer coefficients and if the congruence $f(x) \equiv 0 \pmod p$ has more than n solution, where p is prime, then every coefficient of f is divisible by p.

PROOF : Suppose the theorem is not true. Let c_k be the one coefficient with largest index which is not divisible by p i.e. $p \nmid c_k$. $k < n$. Then the congruence

$$c_o + c_1 x + \ldots + c_k x^k \equiv 0 \pmod p$$

has more than k solution. So, by Largrange's theorem $p \nmid c_k$ which is a contradiction. Hence all the coefficients of f(x) are divisible by p. Hence proved,

THEOREM 18 : For any prime p all the coefficients of the polynomial

$$f(x) = (x-1)(x-2)\ldots(x-p+1) - x^{p-1} + 1 \text{ are divisible by p.}$$

PROOF : Let $g(x) = (x-1)(x-2)\ldots(x-p+1)$ and $h(x)$

$$= x^{p-1} - 1. \text{ Then } f(x) = g(x) - h(x).$$

The roots of g(x) are 1,2...p–1.

Hence g(x) satisfied the congruence $g(x) \equiv 0 \pmod p$.

Now by Euler–Fermat theorem

$$x^{p-1} \equiv 1 \pmod p \Rightarrow x^{p-1} - 1 \equiv 0 \pmod p.$$

That is $h(x) \equiv 0 \pmod p$. It has p–1 roots.

Now $\quad f(x) = g(x) - h(x) \equiv 0 \pmod p$.

By Lagrange's theorem it must have the p–1 number of solutions.

But
$$\begin{aligned} f(x) &= (x-1)(x-2)\ldots(x-p+1) - x^{p-1} + 1 \\ &= x^{p-1} + c_1 x^{p-2} \ldots + (p-1)! - x^{p-1} + 1 \\ &= f_1(x), \end{aligned}$$

where $f_1(x)$ is a polynomial of degree p–2 so f(x) is a polynomial of degree p–2 having p–1 roots implies by theorem 18 that all its coefficients are divisible by p.

THEOREM 19 : (Wilson's theorem) For any prime p we have

$$(p-1)! + 1 \equiv 0 \pmod p \text{ or } (p-1)! \equiv -1 \pmod p.$$

PROOF : Let $f(x) = (x-1)(x-2) \ldots\ldots (x-p+1) - x^{p-1} + 1$

On expanding f(x) we get

$$f(x) = x^{p-1} + c_1 x^{p-2} + \ldots. + c_{p-2} x + c_{p-1} - x^{p-1} + 1$$

where $c_{p-1} + 1$ is the constant term of f(x) that is $c_{p-1} + 1 = (p-1)! + 1$.

Since by theorem 18 all the coefficients of f(x) are divisible by p, hence

$$c_{p-1}+1 = (p-1)! + 1 \equiv 0 \pmod p$$

i.e. $(p-1)! \equiv -1 \pmod p$

which proves Wilson's theorem.

SECOND PROOF OF WILSON'S THEOREM : We can prove this theorem without using Lagrange's theorem.

If p = 2 and p = 3 the theorem is trivial. Since $1! + 1 \equiv 0 \pmod 2$ and $2! + 1 \equiv 0 \pmod 3$ let us take p > 3. Suppose that a is any one of the p–1 positive integers,

$$1,2,3, \ldots p-1$$

and consider the linear congruence $ax \equiv 1 \pmod p$.

Then gcd (a, p) = 1. Hence the linear congruence has unique solution modulo p; hence there is a unique integer a*, with $1 \leq a^* \leq p - 1$, satisfying

$$a\, a^* \equiv 1 \pmod p.$$

Since p is prime a = a* if and only if a = 1 or a = p–1. The congruence

$$a^2 \equiv 1 \pmod p \Leftrightarrow (a-1)(a+1) \equiv 0 \pmod p.$$

Therefore either $a - 1 \equiv 0 \pmod p$ in which case a = 1 or $a + 1 \equiv 0 \pmod p$ in which case a = p– 1.If we will consider a =1 and a = p–1 and group the remaining integers 2,3, .. p–2 into pairs a, a′ where a ≠ a′ such that $a a' \equiv 1 \pmod p$. There are p –3/2 congruences are multiplied together and rearranged we get

$$2.3 \ldots. (p-2) \equiv 1 \pmod p$$

$$\Rightarrow \quad (p-2)! \equiv 1 \pmod p.$$

Multiplying by p–1 we get

$$(p-1)\,(p-2)! \;=(p-1)! \equiv p-1 \equiv -1 \pmod p)$$

i.e. $(p-1)! \equiv -1 \pmod p$

which proves Wilson's theorem.

CONVERSE OF WILSON'S THEOREM :

The converse of Wilson's theorem states that: If $(n-1)! \equiv -1 \pmod n$, then n is a prime.

PROOF : If n is not prime, then n has a divisor d with $1< d < n$. Since $d \le n-1$, d occurs as one of the factors in (n-1)! whence $d \mid (n-1)!$. By hypothesis $n \mid (n-1)! + 1$ since $d \mid n$, these two together imply that $d \mid (n-1)! + 1$.

But $d \mid (n-1)!$ hence $d \mid 1$ which is not true.

Remark : Wilson's theorem and its converse theorem provide us with necessary and sufficient conditions for determining primality of an integer. We say that $n > 1$ is prime if and only if $(n-1)! \equiv -1 \pmod n$. We now apply Wilson's theorem to study the quadratic congruences. $ax^2 + bx + c \equiv 0 \pmod n$ with $(a, n) = 1$.

THEOREM 20 : The quadratic congruences $x^2 + 1 \equiv 0 \pmod p$ where p is a prime, has solutions if and only if $p \equiv 1 \pmod 4$ i.e. $p = 4k + 1$, $k = 1, 2, 3, \ldots$

PROOF : Let a be any solution of $x^2 + 1 \equiv 0 \pmod p$, so that $a^2 = -1$ (mod p).

Since $p \mid a$, Fermat's theorem gives

$$1 \equiv a^{p-1} \equiv (a^2)^{\frac{p-1}{2}} \equiv (-1)^{\frac{p-1}{2}} \pmod p.$$

If $p = 4k + 3$ then R.H.S is –1 and

$$1 \equiv -1 \pmod p \Rightarrow p \mid 2$$

which is impossible. Hence p can not be $4k + 3$.

Therefore p is of the form $4k + 1$.

Converse part

Now

$$(p-1)! = 1.2\ldots..(p-1/2)\,(p+1/2)\,..\,(p-2)\,(p-1)$$

we have

$$(p-1) \equiv -1 \pmod p$$

$$(p-2) \equiv -2 \pmod p$$

$$(p+1)/2 \equiv (p-1)/2 \pmod p$$

Rearranging the factors, we get

$$(p-1)! \equiv 1\ (-1)\ (-2) \ldots ((p-1)/2\,)\ (-(p-1)/2)\ (\text{mod } p)$$

$$\equiv (-1)^{\frac{p-1}{2}}\ (1.2\ldots (p-1)/2)\ (\text{mod } p)$$

since there are (p–1)/2 minus sign is involved.

By Wilson's theorem

$$(p-1)! \equiv -1\ (\text{mod } p).$$

Putting this value we get

$$-1 \equiv (-1)^{\frac{p-1}{2}}\ [\,(p-1)/2\ !]^2\ (\text{mod } p).$$

If $p = 4k + 1$ then$(-1)^{\frac{p-1}{2}} = 1.$

Hence $-1 \equiv [\,(p-1)/2!\,]^2\ (\text{mod } p)$

i.e. $[(p-1)/2]!$ satisfying the quadratic congruence

$$x^2 + 1 \equiv 0\ (\text{mod } p)\ .$$

Example 1 : $p = 17$ p is of the form $4k + 1$. $(p-1)/2 = 8$ and $8! \equiv 40320 \equiv 13\ (\text{mod } 17)$,

Hence $13^2 + 1 \equiv 0\ (\text{mod } 17)$.

Example 2 : If p is odd , p > 1, prove that $1^2 . 3^2 . 5^2 \ldots . (p-2)^2 \equiv (-1)^{p+12}$ (mod p)

and $2^2 . 4^2 . 6^2 \ldots (p-1)^2 \equiv (-1)^{p+1/2}$ (mod p)

Example 1 gives that if p is an odd prime and if q = p–1/2 then

$$(\,(p-1/2)!)^2 + (-1)^{\frac{p-1}{2}} \equiv 0\ (\text{mod } p) \text{ if } p = 4k + 1.$$

Since $k \equiv -(p-k)\ (\text{mod } p)$ it follows that

$$2\ .4.6..\ (p-1) \equiv (-1)^{\frac{p-1}{2}}\ 1.3.5\ldots (p-2)\ (\text{mod } p).$$

Squaring we get,

$$2^2\ 4^2\ 6^2 \ldots (p-1)^2 \equiv (-1)^{p-1}\ 1^{2.}\ 3^2\ 5^2 \ldots (p-2)^2\ (\text{mod } p).$$

Since p is odd p+1/2 is even and p–1 is even,

hence $2^2\ 4^2\ 6^2 \ldots (p-1)^2 = 1^2 .\ 3^2 .\ 5^2\ (p-2)^2 \equiv (-1)^{p+1/2}$ (mod p).

Excercise : For a prime p of the form 4k +3, prove that either

(p–1/2)! ≡ 1 (mod p) or (p–1/2)! ≡ –1 (mod p) hence (p–1/2) ! satisfies the quadratic congruence $x^2 \equiv 1$ (mod p).

Solution : By Wilson's Theorem $-1 \equiv (p-1)! \equiv (-1)^{\frac{p-1}{2}} [\,(p-1/2)!]^2$ (mod p).

$$\equiv (-1)^{2k+1} [\,(p-1/2)!]^2 \text{ (mod p)}$$

(since $p = 4k + 3$)

$$\equiv -[\,(p-1/2)!]^2 \text{ mod p.}$$

$$\Rightarrow \quad [(p-1/2)!]^2 \equiv 1 \text{ (mod p)}$$

$$\Rightarrow \quad (\,(p-1/2)! + 1)\,(\,(p-1/2)\,! - 1) \equiv 0 \text{ (mod p)}$$

$$\Rightarrow \quad (p-1/2)\,! \equiv -1 \text{ (mod p) or } (p-1/2)\,! \equiv 1 \text{ (mod p).}$$

Exercise : Find all positive integers n for which $(n-1)! + 1$ is a power of n.

Solution : If $n = p$ then the congruence $(p-1)! + 1 = 0 \pmod{p^2}$ is true for $p = 5$, $p = 13$, $p = 563$. But no other value of $p < 200000$.

If n is a composite then $(n-1)\,! + 1 = 0 \pmod{n^2}$ is not true.

For example take $n = 6$

$$36 \nmid 5\,! + 1.$$

3.7. SIMULTANEOUS LINEAR CONGRUENCES, THE CHINESE REMAINDER THEOREM AND ITS APPLICATIONS

THEOREM 21 : Let $m_1\ m_2 \ldots m_r$ be pairwise relatively prime integers and $M = m_1\ m_2 \ldots m_r$. Then for integers $b_1, b_2 \ldots b_r$ the system of congruences

$$x \equiv b_1 \pmod{m_1}$$
$$x \equiv b_2 \pmod{m_2}$$
$$x \equiv b_k \pmod{m_k}$$
$$x \equiv b_r \pmod{m_r}$$

has a simultaneous solution which is unique modulo M.

PROOF : $M = m_1\ m_2 \ldots m_r$.

Let $M_k = M/m_k = m_1\ m_2 \ldots m_{k\ 1}\ m_{k+1} \ldots m_r$. Then $(M_k, m_k) = 1$ so that M_k has a unique reciprocal M_k' modulo m_k.

That is $M_k\ M_k' \equiv 1$ (Mod m_k).

Now let $x = b_1\ M_1\ M_1' + b_2\ M_2\ M_2' + \ldots\ldots + b_r\ M_r\ M_r'$.

Consider each term in this sum modulo m_k. Since

$$M_i \equiv 0 \pmod{m_k} \text{ if } i \neq k$$

we have $$x \equiv b_k M_k M_k' \equiv b_k \pmod{m_k}.$$

Hence x satisfies every congruence in the system. But it is easy to show that the system has only one solution mod m. In fact, if x and y are two solutions of the system we have $x \equiv y \pmod{m_k}$ for each k and since the m's are relatively in pairs, we also have $x \equiv y \pmod{M}$. This completes the proof.

THEOREM 22 : Let $m_1, m_2, \ldots m_r$ be relatively prime in pairs. Let $b_1, b_2 .. b_r$ be arbitrary integers and let $a_1, a_2, \ldots a_r$ satisfy $(a_k, m_k) = 1$, for $k = 1, 2, \ldots r$. Then the linear system of congruences

$$\begin{aligned} &a_1 x \equiv b_1 \pmod{m_1} \\ (8) \qquad &a_2 x \equiv b_2 \pmod{m_2} \\ &a_r x \equiv b_r \pmod{m_r} \end{aligned}$$

has exactly one solution modulo $M = m_1 m_2 \ldots m_r$.

PROOF : Since $(a_k, m_k) = 1$ for $k = 1, 2, \ldots r$,

then there exist a_k' such that $a_k a_k' \equiv 1 \pmod{m_k}$ and a_k' is the reciprocal of a_k.

Then the linear congruence

$$a_k x \equiv b_k \pmod{m_k} \Leftrightarrow x \equiv b_k a_k' \pmod{m_k}.$$

Now we apply theorem 15 to the system of congruences

$$x \equiv b_k a_k' \pmod{m_k}, k = 1, 2, \ldots .r.$$

which has a unique solution modulo M i.e. $x = b_k a_k' M_k M_k' \equiv b_k a_k' \pmod{m_k}$

which satisfy the given system of linear congruences.

Example 1 : Find all x which simultaneously satisfy system of congruences

$$x \equiv 1 \pmod 3,$$
$$x \equiv 2 \pmod 4$$
$$x \equiv 3 \pmod 5$$

Solution : $m_1 = 3, m_2 = 4, m_3 = 5$

$$M = m_1 m_2 m_3 = 60 \; M_1 = 60/3 = 20$$

$$M_2 = 60/4 = 15, M_3 = 60/5 = 12.$$

To find the reciprocals of M_1 mod m_1, M_2 mod m_2 and M_3 mod m_3.

Since $$20 \times 2 \equiv 1 \pmod 3 \Rightarrow M_1^1 = 2$$

$$15\text{x}3 \equiv 1 \pmod 4 \Rightarrow M_2^1 = 3$$

$$12\text{x}3 \equiv 1 \pmod 5 \Rightarrow M_3^1 = 3.$$

Hence $x \equiv 1x20x2 + 2x15x3 + 3x12x3 \equiv 238 \pmod{60}$

$\equiv 58 \pmod{60}$.

So the unique solution is 58 (mod 60). We can verify that 58 satisfies the three linear congruences.

Example 2 : Determine the smallest positive integer that gives a remainder of 2 upon division by 3, a remainder of 1 upon division by 5, and a remainder of 6 upon division by 7.

Solution : Let x be a solution. Then the condition requires that

$$x \equiv 2 \pmod 5$$

$$x \equiv 1 \pmod 5$$

$$x \equiv 6 \pmod 7$$

$$M = 3 \times 5 \times 7 = 105.$$

$$M_1 = 35\ ,\ M_2 = 21 \text{ and } M_3 = 15.$$

Now we will determine the inverses of M_i, i = 1, 2, 3,

Now

$$35 \times 2 \equiv 1 \pmod 3 \Rightarrow M_1' = 2$$

$$21 \times 1 \equiv 1 \pmod 5 \Rightarrow M_2' = 1.$$

$$15 \times 1 \equiv 1 \pmod 7 \Rightarrow M_3' = 1.$$

Therefore

$$x \equiv 2.35.2.+ 1.21.1 + 6.15.1 \pmod{105}$$

$$\equiv 140 + 21 + 90 \pmod{105}$$

$$\equiv 251 \pmod{105}$$

$$\equiv 41 \pmod{105}.$$

One can easily verify that 41 is the smallest integer which satisfy the above linear congruences.

APPLICATION OF CHINESE REMAINDER THEOREM :

By applying Chinese remainder theorem we will solve polynomial congruences with composite moduli.

THEOREM 23 : Let f be a polynomial with integer coefficients, let m_1, m_2, ... m_r be positive integer relatively prime in pairs and let $m = m_1 m_2 \ldots m_r$. Then the congruence

(9) $f(x) \equiv 0 \pmod m$

has a solution if, and only if, each of the congruence.

(10) $f(x) \equiv 0 \pmod{m_i}$, $i = 1, 2, \ldots r$.

has a solution. Moreover, if $\nu(m)$ and $\nu(m_i)$ denote the number of solution of (9) and (10) respectively, then

(11)$\nu(m) = \nu(m_1)\ \nu(m_2)\ \ldots.\nu(m_r)$.

PROOF : If $f(a) \equiv 0 \pmod m$ then $f(a) \equiv 0 \pmod{m_i}$ for each $i = 1, 2,\ldots r$. Hence every solution of (9) is a solution of (10).

Conversely, let a_i be a solution (10). Then by the Chinese remainder theorem there exists an integer a such that

(12) $$a \equiv a_i \pmod{m_i} \text{ for } i = 1, 2\ldots r$$

So $$f(a) \equiv f(a_i) \equiv 0 \pmod{m_i}.$$

Since m_i's are relatively prime in pairs we have $f(a) \equiv 0 \pmod m$. Therefore the number of integer a which is a solution of (10) is also a solution of (9). Since each a_i runs through the $\nu(m_i)$ solution of (10) the number of integers a which satisfy (12) and hence (10) is $\nu(m_1)\ldots..\nu(m_r)$. This proves (11).

THEOREM 24 : An even perfect number n ends in the digit 6 or 8 ie

$$n \equiv 6 \pmod{10}$$

or $$n \equiv 8 \pmod{10}.$$

PROOF : A perfect number n is of the form $n = 2^{k-1}(2^k-1)$, where 2^k-1 is a prime k must be prime. If $k = 2$, then $n = 6$. We may confine our attention to the case $k > 2$. $k = 4m + 1$ or $4m + 3$ If $k = 4m + 1$ then

$$\begin{aligned} n &= 2^{4m}(2^{4m+1} - 1) \\ &= 2^{8m+1} - 2^{4m} \\ &= 2.\ 16^{2m} - 16^m \end{aligned}$$

But $$16 \equiv 6 \bmod 10.$$

Hence $$n \equiv 2.6-6 \equiv 6 \pmod{10}.$$

If $$k = 4m+3$$

$$\begin{aligned} n &= 2^{4m+2}(2^{4m+3}-1) \\ &= 2^{8m+5} - 2^{4m+2} \\ &= 2.16^{2m+1} - 4.16^m \end{aligned}$$

Since $$16 \equiv 6 \pmod{10}$$

$$\begin{aligned} n &\equiv 2.6.- 4.6 \pmod{10} \\ &\equiv 2.12 \pmod{10} \\ &\equiv 8 \pmod{10}. \end{aligned}$$

Hence proved.

Exercise : If p is any prime other than 2 or 5, p divides infinitely many

(*i*) 1, 11, 111, 1111, ... 1111111111

(*ii*) 9, 99, 999, 9999, ... 9999999.

Solution :

(*i*)

$$1 = 4-3 = 4k-3$$

$$11 = 4x2+3 = 4k+3$$

$$111 = 4x\ 27+3 = 4k+3$$

$$111\ldots 11111 = 111 \ldots\ldots 108+3 = 4k+3.$$

So these numbers are divisible by primes of the form 4k + 3 and 2 or 5 are not of the form 4k + 3.

(*ii*)

$$9 = 4x\ 2+1$$

$$99 = 4x25-1$$

$$999 = 4x250-1.$$

9, 99, ... are either of the form 4k +1 or 4k–1. So they are divisible by primes of the form 4k+1 or 4k–1 not by 2 or 5.

3.8. SOME ALGORITHMS

ALGORITHM (Chinese Remainder Theorem)

Given pair wise coprime integers m_i ($1 \le i \le k$) and integer x_i, this algorithm finds an integer x such that $x \equiv x_i \pmod{m_i}$ for all i.

1. [Initialize] set $i \leftarrow 1$, $m \leftarrow m_1$, $x \leftarrow x_1$.

2.[finished?] If i = k output x and terminate the algorithm. Otherwise set $i \leftarrow i+1$, and by a suitable version of Euclid's extended algorithm compute u and v such that $u\,m + vm_i = 1$.

3. [Compute next x] set $x \leftarrow umx_i + vm_i\,x$, $m \leftarrow mm_i$, $x \leftarrow x \bmod m$ and go to step 2.

Here we give the following computer program in BASIC to solve the following congruences:

$$x \equiv 0 \pmod{5}$$
$$x \equiv 1 \pmod{7}$$
$$x \equiv 8 \pmod{9}.$$

```
100 REM CHINESE REMAINDER THEOREM
101 PRINT " I WANT YOU TO THINK A NUMBER"
```

```
102 PRINT " LESS THAN 316 WRITE THIS NUMBER"
103 PRINT " DOWN AND DIVIDE BY 5 NOW GIVE ME "
104 PRINT " THE REMAINDER LEFT OVER"
110 INPUT R5
111 PRINT
120 PRINT "NOW DIVIDE YOUR ORIGINAL NUMBER BY"
121 PRINT " 7 AND GIVE ME THIS REMAINDER"
130 INPUT R7
131 PRINT
140 PRINT " NOW DIVIDE YOUR ORIGINAL NUMBER BY"
141 PRINT "9 AND GIVE ME THIS REMAINDER"
150 INPUT R9
151 PRINT
160 REM CALCULATE NUMBER
170 LET A = 126 * R5 + 225 * R7
180 LET X = A – INT (A/315) * 315
190 PRINT
200 PRINT " I AM HAPPY TO TELL YOU THAT YOUR"
201 PRINT "NUMBER CHOSEN WAS", x
210 END
```

RUN

```
I WANT YOU THINK OF A NUMBER
LESS THAN 316 WRITE THIS
NUMBER DOWN AND DIVIDE BY 5.
NOW GIVE ME THE REMAINDER LEFT OVER ? 0
NOW DIVIDE YOUR ORIGINAL NUMBER BY 7
AND GIVE ME THIS REMAINDER ? 1
NOW DIVIDE YOUR ORIGINAL NUMBER BY 9
AND GIVE ME THIS REMAINDER ? 8
I AM HAPPY TO TELL YOU
THAT YOUR NUMBER CHOSE N WAS 260.
```

EXERCISE

1. Find the remainders when 2^{50} and 41^{65} are divided by 7.
2. Prove that the integer $53^{103} + 103^{53}$ is divisible by 39 and that $111^{333} + 333^{111}$ is not divisible by 7.
3. If p is a prime satisfying $n < p < 2n$, show that $\binom{2n}{n} \equiv 0 \pmod{p}$.
4. For $n \geq 1$, using congruence theory establish each of the following:

   (*a*) $7 \mid 5^{2n} + 3.2^{5n\ 2}$

   (*b*) $13 \mid 3^{n+2} + 4^{2n+1}$

   (*c*) $27 \mid 2^{5n+1} + 5^{n+2}$

   (*d*)$43 \mid 6^{n+2} + 7^{2n+1}$
5. Find the remainder when 4444^{4444} is divided by 9,
6. Determine the last three digits of the number 7^{999} (Hints: $7^{4n} \equiv (1 + 400)^n \equiv 1 + 400$ (modulo 100)
7. Find the remainder when 3^{100} is divided by 5.
8. Solve the linear congruence

   (*a*) $36x \equiv 8 \pmod{102}$

   (*b*) $34x \equiv 60 \pmod{98}$

   (*c*) $140x \equiv 133 \pmod{301}$

   (*d*) $6x \equiv 15 \pmod{21}$
9. Solve each of the following set of simultaneous equation

   (*a*) $x \equiv 5 \pmod{11}$ $x \equiv 15 \pmod{31}$, $x \equiv 14 \pmod{129}$

   (*b*) $x \equiv 5 \pmod{6}$, $x \equiv 4 \pmod{11}$, $x \equiv 3 \pmod{17}$

   (*c*) $x \equiv 1 \pmod{3}$, $x \equiv 2 \pmod{5}$, $x \equiv 3 \pmod{7}$

10 Prove that $1835^{1910} + 1986^{2061} \equiv 0 \pmod{7}$.

11 Find the remainder when 2 (26!) is divided by 29.

12 Prove that the odd prime divisors of the integer $n^2 + 1$ are of the form $4k + 1$.

13 Verify that 4 (29!) + 5! is divisible by 31.

14 Show that if $p = 4k + 1$ is prime then $2[(2k)!]^2 \equiv -1 \pmod{p}$.

15 Write a computer program to solve a set of simultaneous congruences when the moduli are relatively prime.

16 Solve $x^2 \equiv 2 \pmod{7^2}$.

17 Show that $2^{50} + 3^{50}$ is divisible by 13.

18 Show that $7 \mid n^2 + 1$ for any n.

19 Determine the remainder when 2^{372} is divided by 37.

20 Show that $11^{81} - 5^{81}$ is divisible by 7.

21 Prove that each member of the set of n–1 consecutive integer $n! + 2$, $n! + 3, \ldots. n! + n$ is divisible by a prime which does not divide any other member of the set.

22. Let a, b, n be positive integer such that n divide $a^n - b^n$. Prove that n also divides $(a^n - b^n) / (a - b)$.

23. Prove that $5 n^3 + 7n^5 \equiv 0 \pmod{12}$ for all integer n.

24. A number whose only digits are 1 is called a Repunit.

Factorize 111111111 (9–one) into product of primes.

25. Find all integral solutions of $x^2 + 1 \equiv 0 \pmod{5^2}$.

26. Find the remainder when $2^{1000000}$ is divided by 77.

27. Prove that $1 + a + a^2 + \ldots. + a^{\varphi(m)-1} \equiv 0 \pmod{m}$

if g.c.d $(a, m) = 1$ and g.c.d $(a-1, m) = 1$.

28. Let $\Phi n = 2^{2^n} + 1$.

(*a*) Prove that $\Phi n \mid (2^{2^{n+1}} - 1)$.

(*b*) Prove that if $a \mid b$, then $(2^a - 1) \mid (2^b - 1)$.

(*c*) Prove that $(2^{2^{n+1}} - 1) \mid (2^{2^{2n}} - 1)$.

29. Prove that if p denotes an odd prime, then $2^{(p-1)/2} \equiv \pm 1 \pmod{p}$.

30. Given any positive integer k, prove that there are k consecutive integers each divisible by a square > 1.

◆◆◆

CHAPTER

4

SOME THEOREMS ON DISTRIBUTION OF PRIME NUMBERS

4.1. INTRODUCTION :

If $x > 0$, we define $\pi(x)$ as the number of primes not exceeding x. Then $\pi(x) \to \infty$ as $x \to \infty$ since there are infinitely many primes. The behaviour of $\pi(x)$ as a function of x has been the object of intense study by many celebrated mathematicians ever since the eighteenth century. Gauss (1792) and Legendre (1798) conjectured that $\pi(x)$ is asymptotic to $x / \log x$, that is $\pi(x) \sim x /\log x$, which means

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1.$$

This conjecture is known as PRIME NUMBER THEOREM (PNT).

The proof of this PNT is given by Hadamard, and de la Vallee Pousin and are analytic in nature. In this chapter we will concern with elementary theorems on primes. We will show that PNT can be expressed in several equivalent forms. We will define in the section 4.2 some auxillary functions which help in establishing equivalent theorems of Prime Number Theorem.

4.2. CHEBYSHEV'S FUNCTION ψ (x) and ϑ (x).

DEFINITION 1 : For $x > 0$ we define Chebyshev's ψ function by the formula

$$\psi(x) = \sum_{n\le x} \Lambda(n),$$

where $\Lambda(n)$ is the Mangoldt function.

Since $\Lambda(n) = 0$, unless n is a prime power we can write the definition of $\psi(x)$ as

$$(1)\qquad \psi(x) = \sum_{n\le x} \Lambda(n) = \sum_{m=1}^{\infty} \sum_{p^{mP}\le x}^{\infty} \Lambda(p^m) = \sum_{m=1}^{\infty} \sum_{p\le x^{1/m}}$$

follows:

The sum on the right of (1) is a finite sum .

For if $x^{1/m} < 2$ i.e. if $1/m \log x < \log 2$

or if $m > \log x / \log 2$

i.e. $\quad m = [\log x / \log 2] + 1$ the sum on p is zero .

$$(2)\qquad \psi(x) = \sum_{m\le \log x/\log 2}^{\infty} \sum_{p\le x^{1/m}} \log p.$$

DEFINITION 2 : if $x > 0$ we define Chebyshev's ϑ – function by the equation

$$(3)\qquad \vartheta(x) = \sum_{p\le x}^{\infty} \log p$$

where p runs over all prime $\le x$.

By using (3) in (2) we get

$$\psi(x) = \sum_{m\le \log x/\log 2} \sum_{p\le \log x^{1/m}} \log p$$

$$= \sum_{m\le \log x/\log 2} \vartheta(x^{1/m})$$

$$= \vartheta(x) + \vartheta(x^{1/2}) + \vartheta(x^{1/3}) + \ldots.$$

$$(4)\qquad \Rightarrow \psi(x) - \vartheta(x) \ge 0.$$

The next theorem gives the relation between $\psi(x) / x$ and $\vartheta(x) / x$.

THEOREM 1 : For $x > 0$ we have

$$0 \le \frac{\Psi(x)}{x} - \frac{\vartheta(x)}{x} \le \frac{(\log x)^2}{2\sqrt{x}\log 2}.$$

Further,

$$\lim_{x\to\infty} \frac{\Psi(x)}{x} - \frac{\vartheta(x)}{x} = 0.$$

PROOF : From (4) we find

$$0 \leq \psi(x) - \vartheta(x) \leq \sum_{2 \leq m \leq \log x/\log 2} \vartheta(x^{1/m}).$$

But

$$\vartheta(x) = \sum_{p \leq x} \log p \leq \sum_{p \leq x} \log x \leq x \log(x).$$

Hence $\vartheta(x^{1/m}) \leq x^{1/m} \log x^{1/m}$.

So

$$0 \leq \psi(x) - \vartheta(x) \leq \sum_{2 \leq m \leq \log x/\log 2} \vartheta(x^{1/m}) \log(x^{1/m})$$

$$\leq \frac{\log x}{\log 2} \sqrt{x} \log \sqrt{x}$$

$$= \frac{\log x}{\log 2} \frac{\sqrt{x}}{2} \log x$$

$$= \frac{\sqrt{x}(\log x)^2}{2 \log 2}.$$

Since x > 0 dividing by x we get

$$0 \leq \frac{\Psi(x)}{x} - \frac{\vartheta(x)}{x} \leq \frac{\sqrt{x}(\log x)^2}{\chi\, 2\log 2} = \frac{(\log x)^2}{\sqrt{x}\, 2\log 2}.$$

Taking limit $x \to \infty$, R.H.S $\to 0$.

Hence $\lim\limits_{x \to \infty} \left(\frac{\Psi(x)}{x} - \frac{\vartheta(x)}{x} \right) = O.$

This implies

$$\lim_{x \to \infty} \psi(x)/x = \lim_{x \to \infty} \vartheta(x)/x.$$

The next theorem plays a key role in establishing relation between $\pi(x)$ and $\psi(x)$ and $\vartheta(x)$ and also establishing equivalent theorems of PNT.

THEOREM 2 : (Abel's identity) For any arithmetical function a(n) let

$$A(x) = \sum_{n \leq x} a(n)$$

where $A(x) = 0$ if $x < 1$.

Assume that f has a continuous derivative on the interval [y, x], where $0 < y < x$, then we have

$$(5) \quad \sum_{y<n\leq x} a(n)\ f(n) = A(x)\ f(x) - A(y)\ f(y) - \int_y^x A(t)\ f'(t)\ dt.$$

PROOF : Let k = [x] and m = [y], [x] stand for greatest integer function ie the greatest integer contained in x. We write x = [x] + {x} where {x} is the fractional part of x.

So $$A(x) = A(k) \text{ and } A(y) = A(m).$$

then $$\sum_{y<n\leq x} a(n)\ f(n) = \sum_{n=[y]+1}^{[x]} a(n)\ f(n)$$

$$= \sum_{n=m+1}^{k} \{A(n) - A(n-1)\}\ f(n)$$

$$= \sum_{n=m+1}^{k} A(n)\ f(n) - \sum_{n=m+1}^{k} A(n-1)\ f(n),$$

Changing the variable in the second term we get

$$\sum_{y<n\leq x} a(n)\ f(n) = \sum_{n=m+1}^{k} A(n)\ f(n) - \sum_{n=m}^{k-1} A(n)\ f(n+1)$$

$$= \sum_{n=m+1}^{k-1} A(n)\ \{f(n) - f(n+1)\} + A(k)\ f(k) - A(m) f(m+1)$$

$$= -\sum_{n=m+1}^{k-1} A(n) \int_n^{n+1} f'(t)\ dt + A(k)\ f(k) - A(m)\ f(m+1)$$

$$= -\sum_{n=m+1}^{k-1} \int_n^{n+1} A(t)\ f'(t)\ dt + A(k)\ f(k) - A(m)\ f(m+1)$$

(since in [n, n +1], A (t) = A(n))

$$= - \int_{m+1}^{k} A(t)\, f'(t)\, dt + A(x)\, f(x) - \int_{k}^{x} A(t)\, f'(t)\, dt$$

$$= A(y)\, f(y) - \int_{y}^{m+1} A(t)\, f'(t)\, dt.$$

$$= A(x)\, f(x) - A(y)\, f(y) - \int_{y}^{x} A(t)\, f'(t)\, dt$$

Hence proved.

4.3. SOME EQUIVALENT FORMS OF THE PRIME NUMBER THEOREM.

THEOREM 3 : The following relations are equivalent:

(6) $\lim\limits_{x \to \infty} \pi(x) \log x \,/\, x = 1$

$\Leftrightarrow$ (7) $\lim\limits_{x \to \infty} \vartheta(x) \,/\, x = 1$

$\Leftrightarrow$ (8) $\lim\limits_{x \to \infty} \psi(x) \,/\, x = 1.$

PROOF : (7) $\Leftrightarrow$ (8) follows from theorem 1.

To prove (6) $\Leftrightarrow$ (7) we require the following lemmas which relate $\pi(x)$ with $\vartheta(x)$.

LEMMA 1 : For $x \geq 2$, we have

(9) $$\vartheta(x) = \pi(x) \log(x) - \int_{2}^{x} \pi(t)/t \;\, dt$$

and

(10) $$\pi(x) = \hat{\pi}(x) / \log(x) - \int_{2}^{x} (\vartheta(t)/\, t \log^2 t) \;\, dt.$$

PROOF : We prove (9) and (10) by using (5).

Let a(n) denote the characteristic function of the primes; then

$$a(n) = \begin{cases} 1 \text{ if } n \text{ isa prime} \\ \\ 0 \text{ otherwise.} \end{cases}$$

Then we have

$$\pi(x) = \sum_{p \le x} p = \sum_{1<n\le x} a(n)$$

$$\vartheta(x) = \sum_{p\le x} \log p = \sum_{1<n\le x} a(n) \log n$$

Taking $f(x) = \log x$ in (5) by taking y=1 we obtain

$$\vartheta(x) = \sum_{1<n\le x} a(n) \log n$$

$$= \pi(x) \log(x) - \pi(1) \log(1) - \int_1^x \pi(t)/t \; dt,$$

which prove (9) since $\pi(t) = 0$ for $t < 2$.

Next, let $b(n) = a(n) \log n$ and write

$$\pi(x) = \sum_{3/2<n\le x} b(n) \; 1/\log n \quad .$$

$$\vartheta(x) = \sum_{n\le x} b(n) \, .$$

Taking $f(x) = 1/\log x$ in (5) with $y = 3/2$,

we obtain

$$\pi(x) = \vartheta(x)/\log(x) - \vartheta(3/2)/\log(3/2) - \int_1^x (\vartheta(t)/t \log^2 t) \, dt$$

which proves (10) since $\vartheta(t) = 0$ if $t < 2$.

PROOF OF THE THEOREM. We have to prove (6) ⇔ (7).

First to show that (6) ⇒(7) we have from (9),

$$\vartheta(x)/x = \pi(x)\cdot\log(x)/x - \frac{1}{x}\int_2^x \pi(t)/t \; dt.$$

Taking limit as x → ∞ we obtain

$$(6) \Rightarrow \lim_{x\to\infty} \pi(x) \log x \ / x = 1$$

(7) follows provided we show that

$$\lim_{x\to\infty} 1 / x \int_2^x \pi(t)/t \ dt = 0.$$

Now

$(6) \Rightarrow \pi(t) / t \ dt = 0(1/\log t)$

for $t \geq 2$ so

$$1/x \int_2^x \pi(t) / t \ dt = 0(1) \ \frac{1}{x}\int_2^x \left(\frac{dt}{\log t}\right).$$

Now

$$\int_2^x dt / \log t = \int_2^{\sqrt{x}} dt / \log t + \int_{\sqrt{x}}^x dt / \log t$$

$$\leq \frac{\sqrt{x}}{\log 2} + \frac{x - \sqrt{x}}{\log\sqrt{x}}.$$

So

$$1/x \int_2^x dt / \log t \to 0.$$

This proves (6) ⇒ (7).

To show (7) ⇒ (6), we know from (10)

$$\pi(x) \log x \ / x = \vartheta(x) / x + \log(x) / x \int_2^x (\vartheta(t)/ t \log^2 t) \ dt.$$

Taking limit as x → ∞ we obtain,

$$\lim_{x\to\infty} (\pi(x) \log x \ / x) = \lim_{x\to\infty} (\vartheta(x) / x) \ ,$$

$$+ \lim_{x \to \infty} \left(\frac{\log x}{x} \right) \int_2^x \; (\vartheta(t) / t \log^2 t) \;\; dt.$$

In order to show (7) ⇒ (6) it is sufficient to show that

$$\lim_{x \to \infty} \; (\log(x) / x) \int_2^x \; (\vartheta(t) / t \log^2 t) \;\; dt \; = 0.$$

But (7) ⇒ ϑ (t) = 0 (t) so

$$\frac{\log x}{x} \int_2^{\lambda} \frac{\vartheta(t)}{t \log^2 t} dt = 0 \left(\frac{\log x}{x} - \int^x \frac{dt}{\log^2 t} \right).$$

Now

$$\int_2^x \frac{dt}{\log^2 t} = \int_2^{\sqrt{x}} \frac{dt}{\log^2 t} + \int_{\sqrt{x}}^x \frac{dt}{\log^2 t}$$

$$\leq \frac{\sqrt{\lambda}}{\log^2 2} + \frac{x - \sqrt{x}}{\log^2 \sqrt{x}}.$$

$$\leq \frac{\sqrt{x}}{\log^2 2} + \frac{x - \sqrt{x}}{\log^2 \sqrt{x}}.$$

Hence $\quad \log x / x \int_2^x \; dt / \log^2 t \to 0.$

This proves (7) ⇒ (6). So (6) and (7) are equivalent. Again (7) and (8) are equivalent Hence we have (6) ⇔ (7) ⇔ (8). Hence proved.

THEOREM 4 : Let p_n denote the n^{th} prime then the following relations are equivalent:

(11) $\lim_{x \to \infty} \pi(x) \log x \; / \; x \; = \; 1$ and

(12) $\lim_{n \to \infty} p_n \; / \; n \log n \; = \; 1.$

PROOF : To show (11) ⇒ (12).

Assume (11) holds. Taking logarithms we obtain

$$\lim_{x\to\infty} [\log\pi(x) + \log\log x - \log x] = 0$$

$$\Rightarrow \quad \lim_{x\to\infty} [\log x(\log\pi(x)/\log x) + (\log\log x / \log x) - 1] = 0$$

$$\Rightarrow \quad \lim_{x\to\infty} \log\pi(x) / \log x = 1.$$

So we get

$$\lim_{x\to\infty} \frac{\pi(x)\log\pi(x)\log x}{x\log x} = \lim_{x\to\infty} \left(\frac{\pi(x)\log x}{x}\right)\left(\frac{\log\pi(x)}{\log x}\right) = 1.$$

That is

$$\lim_{x\to\infty} \pi(x)\log\pi(x) / x = 1.$$

If $x = p_n$ then $\pi(x) = n$ and $\pi(x)\log\pi(x) = n\log n$

Hence, we get

$$\lim_{x\to\infty} \pi(x)\log\pi(x) / x = \lim_{x\to\infty} n\log n / p_n = 1.$$

Next to show that (12) ⇒ (11).

Assume (12) holds. Given x, define n by the inequalities

$$p_n \le x < p_{n+1}$$

so that $n = \pi(x)$. Dividing by n log n, we get

$$\frac{p_n}{n\log n} \le \frac{x}{n\log n} < \frac{p_{n+1}}{n\log n} = \frac{p_{n+1}}{(n+1)\log(n+1)}\frac{(n+1)\log(n+1)}{n\log n}.$$

Now let n → ∞ and using (12) we get

$$\lim_{n\to\infty} x / n\log n = 1$$

$$\lim_{x\to\infty} x / \pi(x)\log\pi(x) = 1$$

or

$$(12)\ \lim_{x\to\infty} \pi(x)\log\pi(x) / x = 1.$$

Taking logarithm of both sides we obtain

$$\Rightarrow \lim_{x\to\infty} \log\left[\log\pi(x)\left(1+\frac{\log\log\pi(x)}{\log\pi(x)}\right)\right] = 0.$$

Since log π (x) → ∞ as x → ∞ it follows that

$$\lim_{x\to\infty}\left(1+\frac{\log\log\pi(x)}{\log\pi(x)}-\frac{\log x}{\log\pi(x)}\right) = 0.$$

$$(13) \Rightarrow \quad \lim_{x\to\infty} \log x \;/\; \log\pi(x) = 1$$

Now we get

$$\lim_{x\to\infty}\frac{\pi(x)\log x}{x} = \lim_{x\to\infty}\left[\frac{\pi(x)\log x}{x}\frac{\log x}{\log\pi(x)}\right]$$

$$= \lim_{x\to\infty}\left(\frac{\pi(x)\log x}{x}\right)\lim_{x\to\infty}\left(\frac{\log x}{\log\pi(x)}\right) = 1.$$

which proves (11) by (12) and (13).

4.4. INEQUALITIES FOR $\pi(n)$ AND p_n

THEOREM 5 : For every integer $n \geq 2$ we have

(14) $1/6 \;\; n/\log n < \pi(n) < 6 \;\; n/\log n.$

PROOF : Consider the binomial coefficient

$$\binom{2n}{n} = \frac{(2n)!}{n!\,n!}.$$

Now

$$4^n = (2^2)^n = 2^{2n} = (1+1)^{2n}$$

$$= \sum_{k=0}^{2n}\binom{2n}{k} > \binom{2n}{n}.$$

Also

$$2^n < \binom{2n}{n}.$$

For $n = 1$

$$2 = \binom{2}{1}.$$

For $n = 2$

$$2^2 = 4 < \binom{4}{2} = 4\,!\ /\,(2!\,2!)\,.$$

Assume it is true for n. To prove it for (n+1)

$$2^{n+1} = 2 \times 2^n \le 2\binom{2n}{2} = 2(2n)!\ /\,(n!\ n!)$$

$$\le \frac{2(n+1)(2n)!(n+1)}{n!(n+1)n!(n+1)}$$

$$= \frac{(2n+1)(2n)!(2n+1)}{(n+1)!(n+1)!}$$

$$\le \frac{(2n+1)(2n)!(2n+1)}{(n+1)!(n+1)!}$$

$$= \frac{(2n+2)!}{(n+1)!(n+1)!}.$$

Hence $$2^n \le \binom{2n}{2}.$$

is proved by induction on n, so we have

$$(15) \qquad 2^n \le \binom{2n}{2} < 4^n.$$

Taking logarithm of both sides of (15) we get

(16) $n \log 2 \le \log (2n)! - 2 \log n! < n \log 4$. $\log (2n)!$ can be calculated from the Legendre's identity .

$$[x]! = \prod_{p\leq x} p^{\alpha(p)}$$

where [x] is the greatest integer function.

This can be proved as follows :

$$\log[x]! = \sum_{n\leq x} \log n = \sum_{n\leq x} \sum_{d/n} \Lambda(d)$$

$$= \sum_{n\leq x} \Lambda(n)[x/n]$$

$$= \sum_{p\leq x} \sum_{m=1}^{\infty} [x/p^m] \log p$$

$$= \sum_{p\leq x} \alpha(p) \log p$$

where $$\alpha(p) = \sum_{m=1}^{\infty} [x/p^m].$$

Hence $\log n! = \sum_{p\leq n} \alpha(p) \log p$; where $\alpha(p) = \sum_{m=1}^{[\log n/\log p]} [n / p^m]$.

Hence

$$\log (2n)! - 2 \log n!$$

$$= \sum_{p\leq 2n} \sum_{m=1}^{[\log 2n/\log p]} \{[2n/p^m] - 2[n/p^m]\} \log p.$$

Since [2x] – 2[x] is either 0 or 1,

(16) implies

$$n \log 2 \leq \sum_{p\leq 2n} \left(\sum_{m=1}^{[\log 2n / \log p]} 1 \right) \log p$$

$$\leq_{,} \sum_{p\leq 2n}{}_{,}= \log 2n = \pi (2n) \log 2n.$$

This gives us

$$(17)\ \pi (2n) \geq \frac{n\log 2}{\log 2n} = \frac{2n}{\log 2n}\frac{\log 2}{12} > \frac{1}{4}\ \frac{2}{\log 2n}$$

since log 2 > ½ . For odd integers we have

(18) π (2n+1) $\geq \pi$ (2n) >

$$\frac{1}{4}\frac{2n}{\log 2n} > \frac{1}{4}\frac{2n}{(2n+1)}\frac{(2n+1)}{\log(2n+1)} \geq \frac{1}{6}\frac{(2n+1)}{\log(2n+1)}.$$

Since 2n / (2n+1) $\geq$ 2/3.

(17) and (18) imply

$$\pi(n) > 1/6 \quad n / \log n$$

for all n $\geq$ 2 which proves left hand side of (14).

We will prove the right hand side inequality of (14).

$$\log (2n)! - 2 \log n! \geq \sum_{p \leq 2n} \{ [2n / p] - 2 [n/p]\} \log p.$$

For primes p in the interval n < p < 2n, we have [2n/p] – 2 [n/p] = 1

So

$$\log (2n)! - 2 \log n! \geq \sum_{n<p\leq 2n} \log p = \vartheta (2n) - \vartheta(n) .$$

(16) implies

$$\vartheta(2n) - \vartheta(n) < n \log 4$$

If $n = 2^r$ then

$$\vartheta(2^{r+1}) - \vartheta(2^r) < 2^r \log 4 = 2^{r+1} \log 2 .$$

Summing on r = 0, 1, 2, ... k, we find

$$\vartheta(2^{k+1}) < 2^{k+2} \log 2.$$

Now we choose k so that $2^k \leq n < 2^{k+1}$, and we obtain

$\vartheta(n) \leq \vartheta(2^{k+1}) < 2^{k+2} \log 2 \leq 4n \log 2$

But if $0 < \alpha < 1$, we have

$$(\pi(n) - \pi(n^{\alpha})) \log n^{\alpha} < \sum_{n^{\alpha}<p\leq n} \log p \leq \vartheta(n) < 4n \log 2.$$

Hence $\pi(n) < (4n \log 2 / \alpha \log n) + \pi(n^{\alpha}) < (4n \log 2 /\alpha \log n) + n^{\alpha}$

$$= n / \log n \,(4 \log 2 /\alpha + \log n / n^{1-\alpha}).$$

Now if c > 0 and n $\geq$ 1, the function $f(x) = x^{c} \log x$ attains its maximum at $x = e^{1/c}$,

so $n^{-c} \log n \leq 1/(ce)$ for $n \geq 1$.

Taking $\alpha = 2/3$ in the inequality for $\pi(n)$ we find

$$\pi(n) < n/\log n\,(6\log 2 + 3/e) < 6\,(n/\log n).$$

This completes the proof.

4.5. PARTIAL SUMS OF MOBIUS FUNCTION

DEFINITIONS : If $x \geq 1$ we define

$$M(x) = \sum_{n \leq x} \mu(n).$$

The exact order of magnitude of M(x) is not known. Numerical evidence suggests that

$$|M(x)| < \sqrt{x} \text{ if } x > 1.$$

Now we prove that the weaker statement

$$\lim_{x\to\infty} M(x)\ /\ x = 0$$

is equivalent to the prime number theorem. The behaviour of $\mu(x)$ as $x \to \infty$ is not so regular so we define another function H(x) as follows:

DEFINITION : If $x \geq 1$ we define $H(x) = \sum_{n \leq x} \mu(n) \log n.$

This function H(x) has relation with $\mu(x)$ which is obvious from the theorem.

THEOREM 6 : We have

$$(19)\quad \lim_{x\to\infty}\left(\frac{M(x)}{x} - \frac{H(x)}{x\log x}\right) = 0.$$

$$\text{i.e.}\quad \lim_{x\to\infty}\frac{M(x)}{x} = \lim_{x\to\infty}\frac{H(x)}{x\log x}.$$

PROOF : Taking $f(t) = \log t$ in theorem 2 with $y = 1$ we obtain

$$H(x) = \sum_{n \leq x} \mu(n)\log n = M(x)\log x - \int_1^x M(t)/t\ \ dt.$$

Hence if $x > 1$ we have,

$$\frac{M(x)}{x} - \frac{H(x)}{x\log x} = \frac{1}{x\log x}\int_1^x \frac{M(t)}{t}\,dt.$$

Therefore to prove the theorem we must show that

$$(20)\quad \lim_{x\to\infty} 1/x\log x \int_1^x M(t)/t\ dt = 0\ .$$

From the definition of M(x) it follows that M(x) – 0 (x) so

$$\int_1^x M(t)/t\,dt = 0\,(\int_1^x dt) = 0\,(x)$$

Putting the value of the integral in (20) we get

$$\lim_{x\to\infty} 1/x\log x \int_1^x M(t)/t\,dt = \lim_{x\to\infty} (1/x\log x)\,0\,(x) = 0.$$

Hence proved.

THEOREM 7 : The prime number theorem implies

$$\lim_{x\to\infty} M(x)/x = 0$$

PROOF : PNT $\Leftrightarrow \lim_{x\to\infty} \psi(x)/x = 1$ i.e. $\psi(x) \sim x$.

We know

$$\Lambda(n) = -\sum_{d|n} \mu(d)\log d$$

By Mobius inversion formula : The equation $f(n) = \sum_{d|n} g(d)$

$\Leftrightarrow g(n) \sum_{d/n} f(d)\,\mu(n/d)$, we have

$$-\mu(n)\log n = \sum_{d|n} \mu(d)\,\Lambda(n/d)$$

$$\Rightarrow \qquad -\sum_{n\le x} \mu(n)\log n = \sum_{n\le x}\sum_{d|n} \mu(d)\,\Lambda(n/d)$$

$$= \sum_{n\le x} \mu(n)\,\psi(x/n).$$

(Since $\psi(x/n) = \sum_{k\le x/n} \Lambda(k)$).

$$(21) \Rightarrow \quad -H(x) = -\sum_{n\le x} \mu(n)\log n = \sum_{n\le x} \mu(n)\,\psi(x/n)$$

Since $\psi(x) \sim x$, if $\varepsilon > 0$ is given, there is a constant $A > 0$ such that

$$|\psi(x)/x - 1| < \varepsilon \text{ whenever } x \geq A.$$

$$\Rightarrow |\psi(x) - x| < \varepsilon x \text{ whenever } x \geq A.$$

Chose $x > A$ and splitting the sum on the right of (21) into two parts,

$$\sum_{n\leq y} + \sum_{y<n\leq x},$$

where $y = [x/A]$.

i.e.

$$\sum_{n\leq x} \mu(n)\psi(x/n) = \sum_{n\leq y} \mu(n)\psi(x/n) + \sum_{y<n\leq x} \mu(n)\psi(x/n)$$

$$= \Sigma_1 + \Sigma_2 \text{ say}$$

Now

$$\Sigma_1 = \sum_{n\leq y} \mu(n)\psi(x/n) = \sum_{n\leq y} \mu(n)(x/n + \psi(x/n) - x/n).$$

$$= x \sum_{n\leq y} \mu(n)/n + \sum_{n\leq y} \mu(n)(\psi(x/n) - x/n)$$

So

$$|\Sigma_1| = |\sum_{n\leq y} \mu(n)\psi(x/n)| \leq x |\sum_{n\leq y} \mu(n)/n|$$

$$+ \sum_{n\leq y} |\psi(x/n) - x/n|$$

$< x + \varepsilon \sum x/n$ (Since $|\psi(x/n) - x/n| < \varepsilon x/n$ if $n \leq x/A = y$)

$< x + \varepsilon x (1 + \log y)$

$< x + \varepsilon x + \varepsilon x \log x.$

In Σ_2 we have $y < n \leq x$ so $n \geq y + 1$

Hence $x/n \leq x/y+1 < A$

because $y \leq x/A < y + 1$

$x/n < A \Rightarrow \psi(x/n) \leq \psi(A).$

$|\Sigma_2| < x\psi(A)$

Hence $|H(x)| = |\sum_{n\leq x} \mu(n)\psi(x/n)| \leq |\Sigma_1| + |\Sigma_2|$

$< (1+\varepsilon)x + \varepsilon x \log x + x\psi(A)$

$< (2 + \psi(A))x + \varepsilon x \log x$ if $\varepsilon < 1$.

So given any ε such that $0 < \varepsilon < 1$ we have,

$| H(x) | < (2 + \psi(A))\, x + \varepsilon\, x \log x$ if $x > A$

$\Rightarrow | H(x) | / \ x \log x < (2+\psi(A)) / \log x \ + \ \varepsilon.$

Choosing $B > A$ so that $x > B$ implies $(2+ \psi(A)) / \log x < \varepsilon$.

Then for $x > B$ we have

$$| H(x) | / x \log x < 2\,\varepsilon$$

$\Rightarrow H(x) / (x \log x) \to 0$ as $x \to \infty$.

Hence proved.

The following gives the converse of theorem 7.

THEOREM 8 : $M(x) = 0\,(x)$ as $x \to \infty$

$\Rightarrow \psi(x) \sim x$ as $x \to \infty$ i.e. PNT.

The proof is difficult. One can refer the book by Apostol (2).

THEOREM 9 : If

$$A(x) = \sum_{n \le x} \mu(n) / n$$

the relation $A(x) = 0\,(1)$ as $x \to \infty$ i.e. $A(x) \to 0$ as $x \to \infty$ implies PNT.

That is PNT is a consequence of the statement that the series $\sum_{n=1}^{\infty} \mu(n)/ n$ is convergent to the sum 0.

PROOF : To prove the theorem we will show that $A(x) = 0\,(1) \Rightarrow \mu(x) = 0\,(x)$ which is equivalent to PNT. By Abel's identity we have

$$M(x) = \sum_{n \le x} \mu(n) = \sum_{n \le x} \mu(n) / n \ . \ n$$

$$= x\,A(x) - \int_1^x A(t)\, dt$$

$$\Rightarrow \qquad M(x) / x = A(x) - 1/x \int_1^x A(t)\, dt$$

$$\Rightarrow \qquad \lim_{x \to \infty} M(x) / x = \lim_{x \to \infty} A(x) - \lim_{x \to \infty} 1/x \int A(t)\, dt.$$

In order to prove the theorem it is sufficient to show that

(22) $\lim_{x\to\infty} 1/x \int_1^x A(t)\,dt = 0.$

Since $A(x) = 0(1)$ as $x \to \infty$, given $\varepsilon > 0$, there exists s such that $|A(x)| < \varepsilon$ if $x \geq s$ and $|A(x)| \leq 1$ for all $x \geq 1$. We have

$$|1/x \int_1^x A(t)\,dt| \leq |1/x \int_1^s A(t)\,dt| + |1/x \int_s^x A(t)\,dt|$$

$$\leq (s-1)/x + \varepsilon(x-s)/x.$$

Letting x → ∞ we find

$$\lim_{x\to\infty} \sup |1/x \int_1^x A(t)\,dt| \leq \varepsilon$$

and since ε is arbitrary this proves (22).

EXERCISE

1. Prove that the following two relations are equivalent ;

   (a) $\pi(x) = x / \log x + 0(x / \log^2 x)$

   (b) $\vartheta(x) = x + 0(x / \log x)$.

2. If $a > 0$ and $b > 0$ then prove that

   $\pi(ax) / \pi(bx) \sim a/b$ as $x \to \infty$.

3. If $0 < a < b$, there exists an x_0 such that $\pi(ax) < \pi(bx)$ if $x \geq x_0$

4. Prove that for every $n > 1$ there exists n consecutive composite numbers.

5. Let S_n denote the sum of first n primes. Prove that for each n there exists an integer whose square lies between S_n and S_{n+1}.

6. Let S_n denote the n^{th} partial sum of the series $\sum_{r=1}^{\infty} 1 / r(r+1)$. prove that for every integer $k > 1$ there exists m and n such that $S_m - S_n = 1/k$.

7. Prove that for all $x \geq 1$ we have $\sum_{n\leq x} \Lambda(n) / n = \log x + 0(1)$.

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CHAPTER

5

CRYPTOGRAPHY

BASIC NOTIONS : Cryptography is the study of method of sending messages in disguised form so that only the intended recipients can remove the disguise and read the message. The message we want to send is called plaintext and the disguised message is called the ciphertext . The plaintext and ciphertext are written in some alphabet consisting of certain number N of letters. The term "letter" (or "character") can refer not only to the families A–Z, but also to numerals blanks, punctuation marks, or any other symbols that we allow our selves to use when writing the messages . The process of converting a plaintext to a ciphertext is called enciphering or encryption, and the reverse process is called deciphering .

The plaintext and ciphertext are broken up into message units. A message unit might be a single letter, or pair of letters (digraph), a triple of letters (trigraph) , or block of 50 letters An enciphering transformation is a function from plaintext message unit to ciphertext message unit i.e. $f: P \rightarrow C$.

The deciphering transformation is the map f^{-1} which recovers the plaintext from the ciphertext i.e. $C \xrightarrow{f^{-1}} P$.

So

(1) $P \xrightarrow{f} C \xrightarrow{f^{-1}} P$.

Such a set up is called cryptosystem .

5.1. CLASSICAL CRYPTOSYSTEM :

Classically, the making and breaking of secret codes has usually been confined to diplomatic and military practices. With the growing quantity of digital data stored and communicated by electronic data processing systems, organization in both public and commercial sectors have felt the need to protect information from unwanted intrusion .

One of the earliest cryptographic system was used by the great Roman emperor Julius Caesar around 50 BC Caesar wrote to Marcus Cicero using a rudimentary substitution cipher in which each letter of alphabet is replaced by the letter which occurs three places down the alphabet, with the last three letters cycled back to the first three letters. thus Caesar Cipher is given by

(2) P: ABCDEFGHIJKLMNOPQRSTUVWXYZ

C: DEFGHIJKLMNOPQRSTUVWXYZ ABC

Example : Plaintext message P is

(3) CAESAR WAS GREAT

is transformed into ciphertext message i.e. by Caesar cipher (2)

(4) FDHVDU ZDV JUHDV.

The Caesar cipher can be described only using congruence theory

We first give numerical values to the alphabets A–Z as follow.

A	B	C	D	E	F	G	H	I	J	K	L	M
01	02	03	04	05	06	07	08	09	10	11	12	13

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
14	15	16	17	18	19	20	21	22	23	24	25	26

Some authors assign values 0–25 to A – Z. One should follow one of these two sets of values.

If P is the digital equivalent of a plaintext letter and C is the digital equivalent of the corresponding ciphertext letter, then

$$C \equiv P + 3 \pmod{26}.$$

Thus the letter of the message (3) are converted to the equivalents :

(5) 03 01 05 19 01 18 23 01 19 07 18 05 01 20. Using the congruence

$$C \equiv P+3 \pmod{26},$$

this becomes the ciphertext.

(6) 06 04 08 22 04 21 26 04 22 10 21 08 04 23.

To recover the plaintext, the procedure is

$$P \equiv C - 3 \equiv C + 23 \pmod{26}.$$

That is given ciphertext if we add 23 to each of the numerical values and reduce it to modulo 26 we get the numerical equivalent of the plaintext and

That is given ciphertext if we add 23 to each of the numerical values and reduce it to modulo 26 we get the numerical equivalent of the plaintext and thus read the message.

The Caesar cipher is simple and hence insecure .

In conventional Cryptosystem, the sender and receiver jointly have a secret key. The sender uses the key to encrypt the plaintext to be sent, while the receiver uses the same key in order to decrypt the ciphertext obtained. Public key cryptography differs from conventional cryptography in that it uses two keys, an enciphering key K_E and the deciphering key K_D. Cryptoanalysis is the science of decoding the ciphertext without the knowledge of the key.

Example 1 : Suppose we are using 26 letter alphabets A–Z with numerical equivalents 0–25.

Let the letter $P \in \{ 0, 1, .. 25\}$ stands for a plaintext message unit.

Define $f : P \rightarrow C$

$$f(P) = \begin{cases} P+3 \text{ if } x < 23 \\ P-23 \text{ if } x \geq 23. \end{cases}$$

i.e. $f(P) = P+3 \bmod 26$

In general we write

$$C = f(P) \equiv P + b \bmod N.$$

where b is a fixed integer which is the letter alphabet with numerical equivalents 0, 1, 2, ... N–1 . To decipher the ciphertext message unit $C \varepsilon \{ 0, 1, 2, \ldots N-1\}$,

we compute

$$P = f^{-1}(C) = C-b \pmod N)$$

and b is called is key and call f a shift transformation .

As a special case we have Caesar cipher with N = 26 and b = 3. Suppose we want to encipher the word "Book". We first convert to numbers.

01 14 14 10,

then add 3 modulo 26, This transforms to

04 17 17 13,

then translate back to letters "ERRN".

To decipher the message one subtract 3 mod 26. For example the ciphertext "ZKB" gives the plaintext "WHY".

Example 2 : (*Frequency Analysis*)

Suppose we intercept a message which we know was enciphered using a shift transformation on single letter of the 26–letter alphabet, we have to find b. For this we use frequency analysis which means the frequently occurring letter in the English language. We note that "E" is the most frequently occurring letter in the English language so it is reasonable to assume that the most frequently occurring letter in ciphertext is the encryption of E.

Suppose the ciphertext message is "PXCDEXFXDPRA". "X" is the most frequently occurring character in the ciphertext. That means the shift takes "E" = 4 to "X" = 23 i.e. $23 \equiv 4 + b \pmod{26}$, so that $b = 19$ to decipher the message, hen, it remains to subtract 19 from the numerical equivalents of

"PXCDEXFXDPRA"

" PXCDEXFXDPRA" = 15 23 02 03 04 23 05 2303 15 17 0

→ 22 04 09 10 11 04 1204 10 22 24 7

= "WE JKLEMEKWYH"

[Since $-19 \equiv 7 \pmod{26}$ we add 7 mod 26]

Example 3 : If we encipher the ciphertext "FQOCUDEM" assuming 'U' as the frequently occurring character in the ciphertext then the message is "PAYMENOW".

AFFINE TRANSFORMATION : This is a more general type of transformation of (Z/NZ) called an affine map.

$C = a P + b \bmod N$, where a and b are fixed integers they together form the enciphering key $K_E = (a,b)$.

To decipher a message that was enciphered by means of the affin map

$C \equiv aP + b \pmod{N}$ to solve for $P \equiv a^1 C + b^1 \pmod{N}$ where a' is the inverse of a mod N and b is equal to $-a^{-1}$ b. This works only if g.c.d. $(a,N) = 1$. The deciphering key K_D is the pair (a^1,b^1).

Example 4 : Suppose a ciphertext is given written in 26–letter alphabet with A–Z assigned the numerical values 0–25. Suppose we know most frequently occurring letter of ciphertext is "K", and the second most frequently occurring letter is "D". It is reasonable to assume that these are the encryption of "E" and "T" respectively, which are the two most frequently occurring letters in the English language. thus replacing the letter by their numerical equivalents and substituting for P and C in the deciphering formula we get

$$10\,a' + b' \equiv 4 \pmod{26}$$

$$3\,a' + b' \equiv 19 \pmod{26}.$$

Since "E" = 4 "D" = 3 "T" = 19 and "K" = 10.

We have to solve the congruence for a' and b' subtracting we get

$$7a' \equiv 11 \pmod{26}$$

and $\quad a' \equiv 7^{-1}.\,11.$

Since $\quad 7.7^{-1} \equiv 1 \pmod{26} \Rightarrow 7^{-1} = 15$

since $\quad 7.15 \equiv 1 \pmod{26}.$

Hence $\quad a' \equiv 7^{-1}.11 \equiv 15.11 \equiv 165 \equiv 9 \pmod{26}$

By putting the value of a′ in the first congruence we get

$$10 \times 9 + b' \equiv 4 \pmod{26}$$

$\Rightarrow \quad b' \equiv 4 - 90 \pmod{26} \equiv 18 \pmod{26}$

So the message can be deciphered by means of the formula

$$P \equiv 9C + 18 \pmod{26}.$$

DIGRAPH : If our message units are digraphs in 27 letter alphabet consisting of A–Z and a blank, we might first let the blank have numerical equivalent 26 and then label the digraph whose two letters corresponding to x, y ∈ { 0, 1, 2, …. 26}. By the integer

$$27x + y \in \{0, 1, 2 \ldots. 728\}.$$

We consider the individual letters as digits to the base 27 and we view the digraph as a 2–digit integer to that base.

Example 1 : The digraph "No" corresponds to the integer

$$27.13 + 14 = 365$$

since $\quad N = 13$ and $0 = 14$.

TRIGRAPH : If we use trigraphs as our message units, we could label them by integer

$$729\,x + 27\,y + z \in \{0, 1 \ldots. 27^3 - 1 = 19682\}.$$

In general, we can label blocks of k letters in an N–letter alphabet by integers between 0 and $N^k - 1$ by regarding each block as a k–digit integer to the base N.

5.2. PUBLIC KEY CRYPTOGRAPHY

We studied in 5.1 that a cryptosystem consist of a 1to 1 transformation f

$$P \xrightarrow{f} C \xrightarrow{f^{-1}} P$$

We use enciphering key K_E is the pair (a,b) and deciphering key K_D to compute f^{-1}.

In affine cryptosystem family, deciphering is accomplished by an affin map, namely

$$P \equiv a^{-1} C - a^{-1} b \pmod N.$$

Here one would have to allow the possibility of examining a large number of corresponding plaintext ciphertext message units because by the definition of public key system, any user can generate an arbitrary number of plaintext – ciphertext pairs.

The name "public key" means that the information needed to send secret message the enciphering key K_E can be made public information without enabling any one to read the secret message.

Suppose we have some population of users of the cryptosystem, each one of whom wants to be able to receive confidential communications from any one of the other users without a third party being able to decipher the message. Some central office can collect the enciphering key $K_{E,A}$ from each user A and publish all of the keys in a "telephone book". Some one wanting to send a message merely has to look up the enciphering key in this "telephone book" and then use the general enciphering algorithm with the key parameters corresponding to the intended recipient . Only the intended recipient has the matching deciphering key needed to read the message .

We notice that with a public key system it is possible for two parties to initiate secret communication without ever having had any prior contact without having established any prior trust for one another, without exchanging any preliminary information. The information necessary is publicly available . To decipher the message it is not always easy for the public to get the deciphering key.

DEFINITION : We define, a public key cryptosystem is a system with the property that some one who knows only how to encipher can not use the enciphering key to find the deciphering key without a prohibitively lengthy computation .

In other words the enciphering function $f : P \to C$ is easy to compute once the enciphering K_E is known, but it is very hard to compute the inverse function

$f^{-1} : C \to P$. That is the function f is not invertible without some additional information. Such a function f is called a one–way function or a trapdoor function.

COMPARISON OF CLASSICAL CRYPTOSYSTEM AND PUBLIC KEY CRYPTOSYSTEM .

Classical Crypto system	Public key Crypto system
1. Once enciphering information is known, the deciphering information can be implemented in approximately same order of magnitude of as the enciphering transformation.	1. Once enciphering key K_E is known it is hard to find the deciphering key K_D to decipher, the message.
2. It takes a little longer time for decipher because one needs to apply Euclidean algorithm to find an inverse modulo N.	2. Time required is less in comparison to classical cryptosystem.
3. The authentication of the communication is not maintained.	3. The authentication of the communication is maintained. The signature of the message known to the persons can read the message.
4.In this system sendig message is faster to implement.	4. In this system for sending message tend to be slower to implement.
5. Lacks in integrity.	5. It has integrity i.e. the recipient of a message should be able to determine that the message has not been modified in transit.
6. The number of plaintext message units per second that can be transmitted is more.	6. The number of plaintext message unit per second that can be transmitted is less.
7. It is widely put to use .	7. It is not widely put to use.
8. Message would be sent faster	8. Message would not be sent faster.

Public key cryptography has found numerous applications:

1. Paper documents such as checks, stocks, lottery tickets can be authenticated using public key techniques. A document is assigned a unique digital signature based on fiber patterns of the paper and the contents of the document. The signature is encoded with a private key and the resulting cipher is affixed to the document. Any one can verity the authenticity of the document by using an embedded public key.

2. Public key techniques are now used in smart card, digital cash other types of electronic banking and commerce .

5.3. THE RSA CRYPTOSYSTEM

The RSA cryptosystem is one of the most popular public–key cryptosystem. It was invented by R.L. Rivest, A. Shamir and L.M. Adleman in 1978 The success of "RSA" cryptosystem and most popular public key cryptosystem , is based on the tremendous difficulty of factoring.

We now describe how RSA works. Each user first choose two extremely large prime numbers p and q and set $n = pq$, knowing the factorization of n, it is easy to compute $\varphi (n) = (p-1) (q-1) = pq + 1 - p - q = n + 1 - p - q$. Next , the user randomly choose an integer e between 1 and φ (n) such that $\gcd(e, \varphi(n)) =1$. Thus each user A chooses two primes P_A and Q_A and a random number e_A which has no common factor with $(p_A-1) (q_A-1)$. Next , A computes $n_A = p_A q_{A,}$ $\varphi(n_A) = n_A+1 -p_A-q_A$ and the multiplication inverse of e_A modulo $\varphi(n_A)$. $d_{A\ \mathrm{def.}\underset{\rightarrow}{=}} e_A^{-1} \bmod \varphi (n_A)$. Then make public and is placed in telephone directory the enciphering key $K_{E,A} = (n_A, e_A)$ and conceals the deciphering key $K_{D,A}= (n_A, d_A)$. Notice that when n_A is openly revealed, the listed public key does not mention the factors p_A and q_A of n_A.

The enciphering transformation is the map

$$f : Z/n_AZ \rightarrow Z/n_AZ \text{ given by } f (P) = P^{e_A} \pmod{n_A}.$$

The deciphering transformation is the map f from Z/n_AZ to it self given by

$$f (C) = C^{d}A \pmod{n_A}.$$

f and f^{-1} are inverse of each other because of our choice of d_A.

To work with it is assumed that the plaintext number $M < n_A$, where n_A is the enciphering modulus . If the message is too length to be handled as a single number $M < n_A$, then M can be broken up into blocks of digits $M_1, M_2 \ldots.. M_r$ of the appropriate size . Each block would be encrypted separately. We will illustrate RSA system by means of example.

Example 1 : Let us choose N = 26, Let the plaintext consists of trigraphs and ciphertext consist of four–graphs in the usual 26 letter alphabet. we want to send the message "YES" to a user A with enciphering key ($n_A, e_A) = (46927, 39423)$, we first find numerical equivalent of "YES" i.e. Y = 24, E = 4, S = 18 $24.26^2 + 4\ 26+18 = 16346$ and then compute

$$P^{e_A} \pmod{m} = 1634^{39423} \pmod{46927}$$

which is $21166 = 1.26^3 + 5.26^3 + 5.26^2 + 8.26 + 2 =$ "BFIC"

To decipher the message the recipient knows the deciphering key

$$\begin{aligned}(n_A, d_A) &= (46927, 26767) = 26767 \text{ and so compute}\\ C^{d_A} (\text{mod } n_A) &= 21166) (\text{mod } 46927) = 16346\\ &= 24.26^2 + 4.26 + 18 = \text{"YES"}.\end{aligned}$$

The user A generates his key like this first he multiplies the primes P_A = 281 and Q_A = 167 to get n_A. Then he chooses e_A at random so that

$$. \gcd (e_A \, \varphi (n_A)) = 1.$$

Then he chooses d_A such that $e_A \, d_A \equiv 1 \pmod{\varphi (n_A)}$. The number p_A, q_A, d_A remain secret.

Example 2 : To encrypt the message "NO WAY" by RSA system.

First select two small primes $p= 29, q = 53$. Then our enciphering modulus

$$\begin{aligned}n &= pq = 29.53 = 1537,\\ \varphi (n) &= 28.52 = 1457\end{aligned}$$

choose the exponent e = 47 such that

$$\gcd (47, 1457) = 1.$$

Find the inverse of e i.e. $e^{-1} = d$ such that

$$\begin{aligned}&e.\, d && \equiv 1 \pmod{1457}\\ &\text{i.e.} && 47.d \equiv 1 \pmod{1457}\\ &\Rightarrow && d = 31.\end{aligned}$$

Now to encrypt the message M = "NO WAY".

The numerical equivalent of plaintext M in 26 – letter alphabet A–Z with numerical equivalent (1 to 26) M = 141500230125.

As M is large split M into blocks of three digits. The first block, 141, encrypt as the ciphertext number $141^{47} \equiv 658 \pmod{1537}$ on the other hand knowing the recovery exponent d = 31 the recipient would begin to recover the first block of the plaintext number $658^{31} \equiv 141 \pmod{1537}$. The total ciphertext of our message M is 0658 1408 1250 1252.

Remark : This method is most secure in the sense that it is not feasible to recover the plaintext M from the information assumed to the know to a third party, the listed public–key (n_A, e_A). The method is to factorize n_A (huge number) then d can be calculated from $\varphi (n_A) = (p_A-1) (q_A-1)$ and e_A. Factoring a large number is very difficult even in computer.

There are other methods of public key cryptography. They are

1. Discrete log method
2. The ElGamal cryptosystem.
3. The Massey–Omura cryptosystem.
4. The knapsack problem we are not discussing in the present text. However in the last chapter as application we will study Discrete log method and the ElGamal crypto system.

EXERCISES

1. Encrypt the message "NUMBER THEORY" using Caesar cipher.
2. Using the linear cipher $C \equiv 5p + 11 \pmod{26}$, encrypt the message "READ NUMBER THEORY".
3. If $n = pq = 274279$ and $\varphi(n) = 272376$, find prime p and q.
4. If the enciphering key in a RSA system $(n,e) = (3233, 37)$. Find the recovery exponent .
5. Decipher the message TZSVIW JQBVMIJ AL MVOOVI; which was produced using the cipher $C \equiv 3p + 7 \pmod{26}$.
6. Suppose that $n = 10088821$ is a product of two distinct primes, and

   $\varphi(n) = 1008\,2272$. Determine the prime factor of n.

   Hints: $p + q = n - \varphi(n) + 1$

   $p - q = [(p+q)^2 - 4n]^{1/2}$.
7. If the modulus in RSA system is $n = 1146115723$ and encrypting key $e = 67$, compute the deciphering key and decrypt the following cipher and recover the plaintext. 474786165 , 121618407.

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CHAPTER

6

PRIMITIVE ROOTS AND INDICES

6.1. PRIMITIVE ROOTS

We define the order, primitive roots of an integer modulo n which has application in cryptography specially in primality testing.

DEFINITION 1 : Let a and n be integers such that (a, n) = 1. Then the order of a modulo n, denoted by ord $_n$ (a) is the smallest positive integer k such that

(1) $$a^k \equiv 1 \pmod{n}.$$

Example 1 : The order of 3 modulo 5 is 4.

we have
$$\begin{aligned} 3 &\equiv 3 \pmod{5} \\ 3^2 &\equiv 4 \pmod{5} \\ 3^3 &\equiv 2 \pmod{5} \\ 3^4 &\equiv 1 \pmod{5}. \end{aligned}$$

Example 2 : The order of 2 mod 17 is 8 and the order of 5 mod 19 is 9.

$$\begin{aligned} 2 &\equiv 2 \pmod{17} \\ 2^2 &\equiv 4 \pmod{17} \\ 2^3 &\equiv 8 \pmod{17} \\ 2^4 &\equiv 16 \pmod{17} \\ 2^5 &\equiv 15 \pmod{17} \\ 2^6 &\equiv 13 \pmod{17} \\ 2^7 &\equiv 9 \pmod{17} \\ 2^8 &\equiv 1 \pmod{17} \end{aligned}$$

$$\begin{aligned} 5 &\equiv 5 \pmod{19} \\ 5^2 &\equiv 9 \pmod{19} \\ 5^3 &\equiv 11 \pmod{19} \\ 5^4 &\equiv 17 \pmod{19} \\ 5^5 &\equiv 16 \pmod{19} \\ 5^6 &\equiv 7 \pmod{19} \\ 5^7 &\equiv 16 \pmod{19} \\ 5^8 &\equiv 4 \pmod{19} \\ 5^9 &\equiv 1 \pmod{19}. \end{aligned}$$

THEOREM 1 : Given $n \geq 1$ and (a, n) = 1. If a has order k modulo n then

(*i*) $a^i \equiv a^j \pmod{n}$, if, and only if $i \equiv j \pmod{k}$.

(*ii*) $a^i \equiv 1 \pmod{n}$, if and only if $i \equiv 0 \pmod{k}$ that is $k|i$.

In particular $k|\,\varphi(n)$.

PROOF OF THE THEOREM : We will first prove (*i*). If $a^i \equiv a^j \pmod{n}$, then $a^{i-j} \equiv 1 \pmod{n}$ by Euclideans algorithm we write.

$$i \cdot j = qk + r, \text{ where } 0 \leq r < k.$$

Then $1 \equiv a^{i-j} = a^{qk+r} \equiv a^r \pmod{n}$ since a has order k.

So, $r = 0$ and $i \equiv j \pmod{k}$.

Conversely if $i \equiv j \pmod{k}$ then $i - j = qk$.

So $a^{i-j} \equiv a^{qk} \equiv (a^k)^q \equiv 1 \pmod{n}$

Hence $a^i \equiv a^j \pmod{n}$.

Proof of (*iii*) follows from (*i*).

Proof of (*ii*) If $a^i = 1 \pmod{n}$ then $i \geq k$. because k is the order a modulo n,

we write $i = kq + r,\ 0 \leq r < k$.

Now $a^i = a^{kq+r} = a^{kq}\, a^r = (a^k)^q\, a^r$

$$\equiv 1^q\, a^r \equiv a^r \equiv a^r \pmod{n}.$$

So $a^i \equiv 1 \pmod{n}$ implies that $a^r \equiv 1 \pmod{n}$

which is impossible since $0 \leq r < k$

Hence $r = 0$ i.e. $i = kq$.

Hence $k \mid i$.

Conversely if $i \equiv 0 \pmod{k}$ i.e. $k.|\, i$ we write $i = kq$. Then we get

$$a^i = a^{kq} = (a^k)^q \equiv 1 \pmod{n}..$$

By Euler –Fermat's theorem we know, if $(a, n) = 1$,

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

By (*ii*) it follow that $k|\,\varphi(n)$. Since k is the order r of a modulo k. Hence proved.

THEOREM 2 : If the integer a has order k modulo n and $h > 0$. then a^h has order

$k / \gcd(h, k)$.

PROOF : Let $d = \gcd(h, k)$. Then we write $h = h_1 d$ and $k = k_1 d$ with $\gcd(h_1, k_1) = 1$.

Clearly

$$(a^h)^{k_1} = (a^{h,\, d})^{k/d} = (a^k)^{h_1} \equiv 1 \pmod{n}.$$

If a^h is assumed to have order r mod n, then we have $r|k_1$. On the other hand , since a has order k mod n, the congruence

$$a^{hr} \equiv (a^h)^r \equiv 1 \pmod{n}$$

implies that k| hr, in other words $k_1d \mid h_1dr$ or $k_1 |h_1r$. But gcd $(h_1, k_1) = 1$ and therefore $k_1|r$. This gives

$$r = k_1 = k/d = k / \gcd(h, k).$$

This completes the proof of the theorem.

Corollary : Let a has order k modulo n. Then a^h also has order k if and only if

$$\gcd(h,k) = 1.$$

The proof follows directly from the above theorem by taking d = 1.

DEFINITION 2 : (Primitive Root) If the order of a mod n is $\Phi(n)$ then a is called a primitive root of n or a primitive root modulo n.

The statement that a is a primitive root of n implies the following :

(*a*) (a, n) = 1

(*b*) $a^{\varphi(n)} \equiv 1 \pmod{n}$

(*c*) $a^h \equiv 1 \pmod{n}$, $0 < h < \varphi(n)$.

Conversely (*a*), (*b*), (*c*) imply that a is a primitive root of n . It follows from the definition.

Example 1 : Show that 2 is a primitive root of 11.

Solution : If $a^d \equiv 1 \pmod{n}$ then d divides φ (11). The divisor of φ (11) = 10 are 2, 5 and 10.

We then find

$$2^2 \equiv 4 \pmod{11}$$
$$2^5 \equiv -1 \pmod{11}$$
$$2^{10} \equiv 1 \pmod{11}.$$

So the smallest integer x which satisfies $2^x \equiv 1 \pmod{11}$ is 10. This implies 2 is a primitive root of modulo 11.

Example 2 : Show that 5 is a primitive root of 18.

Solution : φ (18) = 6 = 2 x 3. Hence the divisors of φ (18) are 2, 3, and 6.

Now

$$5^2 \equiv 7 \pmod{18},\ 5^3 \equiv -1 \pmod{18},\ 5^6 \equiv 1 \pmod{18}$$

It follow that 5 is a primitive root of 18.

THEOREM 3 : Let (a, n) = 1 then a is a primitive root of if and only if

(2) $a^{\varphi(n)/p} \not\equiv 1 \pmod{n}$

for every prime divisor p of φ(n).

PROOF : Let a be such that (2) holds i.e.

$$a^{\varphi(n)/p} \not\equiv 1 \pmod{p}$$

for every prime divisor p of φ (n). Let us assume that under this condition a is not a primitive root of n. Then it follows that the order a is some integer k less than φ(n). This implies that k divides φ(n). Thus φ(n) / k is an integer and is therefore divided by some prime divisor p of φ (n).

Hence

$$a^{\varphi(n)/p} = (a^k)^{\varphi(n)/kp} \equiv 1 \pmod{n}$$

because $a^k \equiv 1 \pmod{n}$. This contradision (2) alone .

Let a be a primitive root of n. Then it follows that

$$a^{\phi(n)} \equiv 1 \pmod{n} \text{ and } a^h \not\equiv 1 \pmod{n}.$$

This means that $a^{\varphi(n)/p} \not\equiv 1 \pmod{n}$ for every prime divisor p of φ (n).

Example 1 : Find all primitive roots of 19.

Solution : $\varphi(19) = 18 = 2x3^2$. So 2, 3 are only prime divisor of φ(n). But

$$\frac{\varphi(19)}{3} = \frac{18}{3} = 6 \text{ and } \frac{\varphi(19)}{2} = \frac{18}{2} = 9.$$

So , a is a primitive root of 19 if and only if $a^6 \not\equiv 1$ (mod 19) and $a^9 \not\equiv 1$ (mod 19). Letting a = 2, 3, ...18 in succession we find the following modulo 19.

$2^6 \equiv 7$	$2^9 \equiv -1$	Primitive root
$3^6 \equiv 7$	$3^9 \equiv -1$	Primitive root
$4^6 \equiv 11$	$4^9 \equiv 1$	
$5^6 \equiv 7$	$5^9 \equiv 1$	
$6^6 \equiv 11$	$6^9 \equiv 1$	
$7^6 \equiv 1$	$7^9 \equiv 1$	
$8^6 \equiv 1$	$8^9 \equiv -1$	
$9^6 \equiv 11$	$9^9 \equiv 1$	
$10^6 \equiv 11$	$10^9 \equiv -1$	Primtive root
$11^6 \equiv 1$	$11^9 \equiv 1$	

$12^6 \equiv 1$	$12^9 \equiv -1$	
$13^6 \equiv 11$	$13^9 \equiv -1$	Primitive root
$14^6 \equiv 7$	$14^9 \equiv -1$	Primitive root
$15^6 \equiv 11$	$15^9 \equiv -1$	Primitive root
$16^6 \equiv 7$	$16^9 \equiv 1$	
$17^6 \equiv 7$	$17^9 \equiv 1$	
$18^6 \equiv 1$	$18^9 \equiv -1.$	

It follow that 2, 3, 10, 13, 14 and 15 are primitive roots modulo 19.

Exercise : Find all primitive roots of 15, 17.

Example 2 : If $F_n = 2^{2^n} + 1$, $n > 1$ is a prime, then 2 is not a primitive root of F_n.

Solution : Since $2^{2^{n+1}} - 1 = (2^{2^n} + 1)(2^{2^n} - 1)$

we have $\quad 2^{2^{n+1}} \equiv 1 \pmod{F_n}$

which implies that the order of 2 modulo F_n does not exceed 2^{n+1}. But if F_n is assumed to be prime,

$$\varphi(F_n) = F_n - 1 = 2^{2^n}$$

and since $2^{2^n} > 2^{n+1}$, whenever $n > 1$.

Thus the order of 2 modulo F_n is smaller than $\varphi(F_n)$; ie 2 can not be a primitive root of F_n.

But 2 is a primitive root $5 = F_1$.

THEOREM 4 : Let gcd (a, n) = 1 and let $a_1, a_2 .... a_{\phi(n)}$ be the positive integer less than n and relatively prime to n. If a is a primitive root of n, then a, a^2 , $a^{\varphi(n)}$ are congruent modulo n to $a_1, a_2 .... a_{\phi(n)}$ in some order.

PROOF : Since (a, n) = 1 the same is true for all powers of a; hence, each $a^k \equiv a_l \pmod{n}$. The $\varphi(n)$ numbers in the set $\{a, a^2 ... a^{\varphi(n)}\}$ are incongruent modulo n. Hence these powers must represent the integer $a_1, a_2, \ldots a_{\phi(n)}$.

Corollary 1 : If n has a primitive root, then it has exactly $\varphi(\varphi(n))$ of them.

PROOF : Suppose that a is a primitive root of n. By the theorem, any other primitive root of n is found among member of the set $\{a, a^2 \ldots a^{\varphi(n)}\}$. But the number of powers a^k, $1 \le k \le \varphi(n)$ which have order $\varphi(n)$ is equal to the number of integer k for which gcd (k, $\varphi(n)$) = 1; there are $\varphi(\varphi(n))$ such integers, hence $\varphi(\varphi(n))$ primitive roots of n.

Example : Take $a = 2$ and $n = 9$, then

$\varphi(9) = 6$, the first 6 powers of 2 i.e. $2, 2^2, 2^3, 2^4, 2^5, 2^6$ must be congruent modulo 9, in some order to the positive integers less than 9 and relatively prime to it. These integers less than 9 and relatively prime to 9 are 1, 2, 4, 5, 7, 8 and

$$\begin{aligned} 2^1 &\equiv 2 \pmod{9} \\ 2^2 &\equiv 4 \pmod{9} \\ 2^3 &\equiv 8 \pmod{9} \\ 2^4 &\equiv 7 \pmod{9} \\ 2^5 &\equiv 5 \pmod{9} \\ 2^6 &\equiv 1 \pmod{9}. \end{aligned}$$

There are $\varphi\ (\varphi(9) = \varphi\ (6) = 2$, primitive roots of 9. These are the integers 2 and 5.

THEOREM 5 : Let a be a primitive root of n and let $a_1 \equiv a \pmod{n}$. Then a_1 is also a primitive root of n.

PROOF : The order of a (mod n) is $\varphi(n)$. Hence the order of a_1 (mod n) is $\varphi(n)$. This implies that a_1 is a primitive root of n. Hence proved.

6.2. REDUCED RESIDUE SYSTEM.

THEOREM 6 : Let $(a, n) = 1$. Then a is a primitive root modulo n if , and only if the numbers

(3) $a, a^2, \ldots.. a^{\varphi(n)}$

form a reduced residue system modulo n.

PROOF : If a is a primitive root of n the number in (3) are incongruent modulo n since there are $\varphi(n)$ such numbers they form a reduced system modulo n .

Conversely, if the numbers in (3) form a reduced residue system, then

$$a^{\varphi(n)} \equiv 1 \pmod{n}.$$

But no smaller power is congruent to 1, so a is a primitive root. Hence proved .

If n has a primitive root then each reduced residue system modulo n can be expressed as a geometric progression. But the primitive root exists only for the following modulo:

$$n = 1, 2, 2^2, p^k \text{ and } 2p^k$$

where p is an odd prime and $k \geq 1$.

The case $n = 1$ is trivial For $n = 2$, the number 1 is a primitive root.

For n = 4, φ (4) = 2 and $3^2 = 1$ (mod 4), so 3 is a primitive root (mod 4).

We will show that there are now primitive root modulo 2^k it $k \geq 3$.

THEOREM 7 : For $k \geq 3$, the integer 2^k has no primitive root.

PROOF : To show that if a is an odd integer, then for $k > 3$

(4) $$a^{2^{k-2}} \equiv 1 \pmod{2^k}.$$

If k = 3, (4) becomes

$$a^2 \equiv 1 \pmod 8 \text{ which is true for } a = 1, 3, 5, 7.$$

Assume that (4) holds for integer k, i.e. $a^{2^{k-2}} \equiv 1 \pmod{2^k}$ equivalently

$$a^{2^{k-2}} \equiv 1 + m\,2^k \text{ where m is an integer.}$$

Squaring both sides we get

$$\begin{aligned} a^{2^{k-1}} &= (a^{2^{k-2}})^2 = (1 + m2^k)^2 \\ &= 1 + 2\,(m2^k) + (m2^k)^2 \\ &= 1 + 2^{k+1}\,(m + m^2\,2^{k-1}) \\ &\equiv 1 \pmod{2^{k+1}} \end{aligned}$$

So (4) holds for k +1 and hence for all $k \geq 3$.

Now the integers which are relative prime to 2^k are the odd integer; also $\varphi(2^k) = 2^{k-1}$. If a is an odd integer and $k \geq 3$,

$$a^{\varphi(2^k)/2} \equiv 1 \pmod{2^k}$$

and , consequently there are no primitive roots of 2^k.

THEOREM 8 : If gcd (m, n) = 1, where m > 2 and n > 2, then the integer mn has no primitive root.

PROOF : Consider any integer a for which (a, mn) = 1, then gcd (a, m) = 1 and gcd (a, n) = 1, put h = lcm (φ(m), φ (n)) and d = gcd (φ (m), φ (n)).

Since φ (m) and φ (n) are both even, $d \geq 2$.

$$h = \frac{\varphi(m)\,\varphi(n)}{d} \leq \frac{\varphi(mn)}{2} \quad \left(\text{Since l cm}\,(a, b) = \frac{ab}{\gcd(a, b)}\right)$$

By Euler's Theorem $a^{\varphi(m)} \equiv 1 \pmod m$

$$\begin{aligned} a^h &= (a^{\varphi(m)})^{\varphi(n)/d} \equiv (1)^{\varphi(n)/d} \\ &\equiv 1 \pmod m. \end{aligned}$$

Similarly $a^h \equiv 1 \pmod{n}$.

Since $\gcd(m, n) = 1$ These two imply that

$$a^h \equiv 1 \pmod{mn}.$$

Hence there can be no primitive root of mn.

6.3. PRIMITIVE ROOTS MODULO PRIME

We now determine all integers that have primitive roots belonging to them. We prove the following theorem.

THEOREM 9 : There exists at least one primitive root modulo each prime $p \geq 3$.

PROOF : Let p be a prime ≥ 3. To find integer that have order $\varphi(p) = p - 1 \pmod{p}$.

If a is any integer having order n mod p, then $(a, p) = 1$, implies

$$a^{p-1} \equiv 1 \pmod{p} \text{ (Fermat's Theorem) Hence } n \mid p-1.$$

Conversely, for any divisor m of p–1 the poynomial congurence

(5) $x^m \equiv 1 \pmod{p}$

has at most m mutually incongruent solutions. Let b be an integer having order m (mod p). Then b is a solution of (5) . In fact, $1, b, b^2, .. b^{m-1}$ are all the solutions of (5) (modulo p).

The congruence (5) can not have any other solution other than $1, b, b^2, \ldots b^{m-1}$. Hence any integer a having order m (mod p) must be a power of b. If $c = b^r$, c has order m (mod p) $\Leftrightarrow (r, m) = 1$. Thus only $\varphi(m)$ integers among $1, b, b^2 \ldots b^{m-1}$ have order m (mod p).

That is the number of integer having order m (mod p) is $\varphi(m)$, if there exists an integer having order m (mod p).

Let A_s = number of integer k, $1 \leq k \leq p-1$ which have order s (mod p) and if there is no integer having order's (mod p) then

$A_s = (0)$. Since each integer k, $1 \leq k \leq p-1$ has some order m (mod p) then $m \mid p-1$.

Hence

$$p-1 = \sum_{s|p-1} A_s$$

$$p-1 = \sum_{s|p-1} \varphi(s)$$

Hence

(6) $\sum_{s|p-1} A_s = \sum_{s|p-1} \varphi(s) = p-1.$

If $A_s > 0$ then $A_s = \varphi(s)$ otherwise $A_s \leq \varphi(s)$. From (6) we get $A_s = \varphi(s)$ for all s.

$\varphi(p-1) = A_{p-1}$ ie $A_{p-1} = \varphi(p-1)$ implies there exists at least one primitive root (mod p) for each prime $p \geq 3$.

Example 1 : Take p = 17.

$p-1 = 16$ Divisors of 16 are $1, 2, 2^2, 2^3, 2^4$, .

Integer having order 1 (mod 17) is 1 and $\varphi(1) = A_1$,

order 2 (mod 17) is 16 since $2^{16} \equiv 1 \pmod{17}$ and $\varphi(2) = A_2$.

Order 4 (mod 17) is (4, 13) and $\varphi(4) = 2 = A_4$; order 8 (mod 17) is (2, 8, 9, 12) and $\varphi(8) = 4 = A_s$; and order 16 (mod 17) is { 3, 5, 6, 7, 10, 11, 14, 13}

and $\varphi(16) = 8 = A_{16}$.

Thus p = 17 has 8 primitive roots belonging to it.

6.4. THE EXISTENCE OF PRIMITIVE ROOTS MOD p^k.

Consider the case $m = p^k$, where p is an odd prime and $k \geq 2$. To find primitive roots mod p^k. Let a be such a primitive root mod p. To see whether a is primitive root mod p^2. Now $a^{p-1} \equiv 1 \pmod{p}$, and since $\varphi(p^2) = p(p-1) > p-1$, a is not a primitive root mod p^2 if $a^{p-1} \equiv 1 \pmod{p^2}$. Therefore

(7) $a^{p-1} \not\equiv 1 \pmod{p^2}$

is a necessary conditions for a primitive root a mod p to be a primitive root mod p^2 condition (7) is also sufficient for a to be primitive root mod p^2, and also mod p^k for $k \geq 2$ we have the following theorem.

THEOREM 10 : Let p be an odd prime. Then (*i*) if a is a primitive root mod p then a is also a primitive root mod p^k for all $k \geq 1$ if and only if (7) holds.

(*ii*) There exists at least one primitive root a mod p which satisfies (7), hence there exists at least one primitive root mod p^k, if $k \geq 2$.

PROOF : Proof of (*ii*) Let a be a primitive root mod p. If $a^{p-1} \not\equiv 1 \pmod{p^2}$ there is nothing to prove. On the other hand if $a^{p-1} \equiv 1 \pmod{p^2}$ then $a_1 = a + p$ is another primitive root modulo p, satisfying $a_1^{p-1} \not\equiv 1 \pmod{p^2}$ because

$$a_1^{p-1} = (a+p)^{p-1} = a^{p-1} + (p-1)\, a^{p-2}\, p + sp^2$$

$$= a^{p-1} + (p^2 - p)\, a^{p-2} \pmod{p^2}$$

$$\equiv 1 - p\, a^{p-2} \pmod{p^2}.$$

Since the second term can not be divisible by p^2 because

if $\quad p\, a^{p-1} \equiv 0 \pmod{p^2}$

then we have $a^{p-2} \equiv 0 \pmod{p^2}$ contradicting the fact that a is a primitive root mod p.

Hence $\quad a_1{}^{p-1} \not\equiv 1 \pmod{p^2}$.

Next to prove (*i*), let a be a primitive root modulo p. If this a is a primitive root mod p^k for all $k \geq 1$, then it is primitive root mod p^2 which implies (7) is true.

Converse part : suppose that a is a primitive root mod p which satisfies (7). To show that a is a primitive root mod p^k for all $k \geq 2$. Let r be the order of a mod p^k Then to show that $r = \varphi(p^k)$. Since $a^r \equiv 1 \pmod{p^k}$,

we have $\quad a^r \equiv 1 \pmod{p}$ so $\varphi(p) \mid r$. We write

(8) $r = q\, \varphi(p)$.

Nowr$\mid \varphi(p^k)$ so $q\, \varphi(p) \mid \varphi(p^k)$

But $\quad \varphi(p^k) = p^{k-1}(p-1)$.

Hence $\quad q(p-1) \mid p^{k-1}(p-1)$

which means $q \mid p^{k-1}$ Therefore $q = p^s$

where $s \leq k-1$; and (8) becomes $r = p^s(p-1)$.

If we prove that $s = k-1$ then $r = \varphi(p^k)$ and the proof is complete. Suppose it is not then $s \leq k-2$ and we have $r = p^s(p-1) \mid p^{k-2}(p-1)$ i.e. $\varphi(p^{k-1}) = m\, r$.

Thus, since $\varphi(p^{k-1})$ is a multiple of r, this implies

$a^{\varphi(p^{k-1})} \equiv 1$, $a^{mr} \equiv (a^r)^m \equiv 1 \pmod{p^k}$ since $a^r \equiv 1 \pmod{p^k}$ which is a contradiction follows from the following theorem.

THEOREM 11 : Let a be a primitive root modulo p such that

(9) $a^{p-1} \not\equiv 1 \pmod{p^2}$.

Then for every $k \geq 2$ we have

(10) $a^{\varphi(p^{k-1})} \not\equiv 1 \pmod{p^k}$.

PROOF : The proof is by induction on k. For k = 2, (10) reduces to (9). Suppose (10) holds then

$$a^{\varphi(p^{k-1})} \equiv 1 \pmod{p^{k-1}} \text{ (Euler–Fermat's Theorem)}$$

so $$a^{\varphi(p^{k-1})} = 1 + mp^{k-1}$$

where $p \nmid m$. Raising both sides to p^{th} power we get

$$\begin{aligned} a^{\varphi(p^k)} &= (1 + mp^{k-1})^p \\ &= 1 + mp^k + m^2\, p\, (p-1)/2\; p^{2(k-1)} + tp^{3(k-1)} \end{aligned}$$

now $2k - 1 \geq k + 1$ and $3\,k-3 \geq k + 1$ since $k \geq 2$.

Hence we get

$$a^{\varphi(p^k)} \equiv 1 + mp^k \pmod{p^{k+1}}$$

where $p \nmid m$. Hence

$$a^{\varphi(p^k)} \not\equiv 1 \pmod{p^{k+1}}.$$

So (10) holds for $k + 1$ if it holds for k. This completes the proof of the theorem.

6.5. THE EXISTENCE OF PRIMITIVE ROOTS MODULO $2p^k$

THEOREM 12 : If p is an odd prime and $k \geq 1$ there exists odd primitive roots a modulo p^k. Each such a is also a primitive root mod $2p^k$.

PROOF : If a is a primitive root mod p^k so is $a + p^k$ as shown earlier. One of a or $a + p^k$ is odd . Hence odd primitive roots modulo p^k always exist. Let a be an odd primitive root mod p^k and let r be order of a mod $2p^k$. To show that $r = \varphi\,(2p^k)$.

Now r divides $\varphi\,(2p^k)$, and $\varphi\,(2p^k) = \varphi\,(2)\,\varphi\,(p^k)\,\varphi(p^k)$.

So r also divides $\varphi\,(p^k)$. Again $a^r \equiv 1 \pmod{2p^k}$.

so $a^r \equiv 1 \pmod{p^k}$. Hence $\varphi\,(p^k) \mid r$ since a is primitive root modulo p^k. Therefore $r = \varphi\,(p^k) = \varphi\,(2p^k)$, so a is a primitive root mod $2p^k$.

THEOREM 13 : Given $m \geq 1$ where m is not of the form $m = 1, 2, p^k$ or $2p^k$ where p is an odd prime. then for any a with $(a, m) = 1$ we have

$a^{\varphi(m)/2} \equiv 1 \pmod{m}$.

So there are no primitive root mod m.

PROOF : we have shown earlier that there are no primitive roots mod 2^k if $k \geq 3$.

Since $m \geq 1$, m has following factorization :

$$m = 2^k\, p_1{}^{k1} \ldots p_s{}^{ks}.$$

where p_l is are odd primes, $s \geq 1$, and $k \geq 0$. Since m is not of the form 1, 2, p^k or $2p^k$ we have $k > 2$ if $s = 1$ and $s > 2$ if $k = 0$ or 1

Now

$$\varphi(m) = \varphi(2^k)\, \varphi(p_1^{k_1}) \ldots \varphi(p_s^{k_s}).$$

Let (a, m) = 1 To prove that $a^{\varphi(m)/2} \equiv 1 \pmod m$ Let b be a primitive root mod $p_1^{k_1}$ and choose r such that

$$a \equiv b^r \pmod{p_1^{k_1}}.$$

Then we have

(11) $a^{\varphi(m)/2} \equiv b^{r\varphi(m)/2} \equiv b^{t\varphi(p_1k_1)} \pmod{p_1^{k_1}}$

where

$$t = r\,\varphi(2^k)\, \varphi(p_2^{k_2}) \ldots \varphi(p_s^{k_s})/2.$$

If $k \geq 2$ the factor $\varphi(2^k)$ is even and hence t is an integer. If k = 0 or 1 then $s \geq 2$ and the factor $\varphi(p_2^{k_2})$ is even so t is an integer in this case. Hence (11) gives us

$$a^{\varphi(m)/2} \equiv 1 \pmod{p_1^{k_1}}.$$

Similarly we get

(12) $a^{\varphi(m)/2} \equiv 1 \pmod{p_i^{k_i}}$

for each i = 1,2,... s. To show that (12) holds mod 2^k. If $k \geq 3$ the condition

(a, m)= 1 requires a to be odd and we get by previous theorem

$$a^{\varphi(2^k)/2} \equiv 1 \pmod{2^k}.$$

Since $\varphi(2^k) \mid \varphi(m)$ this gives

(13)$a^{\varphi(m)/2} \equiv 1 \pmod{2^k}$.

for $k \geq 3$.

If $k \leq 2$ we have

(14) $a^{\varphi(2^k)} \equiv 1 \pmod{2^k}$.

But $s \geq 1$ so $\varphi(m) = \varphi(2^k)\, \varphi(p_1^{k_1}) \ldots \varphi(p_s^{k_s}) = 2l\, \varphi(2^k)$, where l is an integer Hence $\varphi(2^k) \mid \varphi(m)/2$ and (14) implies (13) for $k \leq 2$. Hence (13) holds for all k. Multiplying (13) and (12) we get $a^{\varphi(m)/2} \equiv 1 \pmod m$ which proves that a can not be a primitive root mod m.

Example 1 : Let p = 5 Find a primitive root mod 5^2.

Solution : We know 2 is a primitive root mod 5. Since $2 \equiv 2 \pmod 5$, $2^2 \equiv 4 \pmod 5$

$$2^3 \equiv 3 \pmod 5,\ 2^4 \equiv 1 \pmod 5.$$

Again $2^4 \equiv 1 \pmod{5^2}$ and $2^{20} \equiv 1 \pmod{5^2}$ 2 is a primitive root $\pmod{5^2}$.

Exercise 1 : Show that 3 is a primitive roots mod 5^2 .

Exercise 2 : Determine all primitive roots of $3^2, 3^3, 3^4, 3^5$.

6.6. INDICES.

DEFINITION : Let a be a primitive root mod m. For any integer b coprime to m if $b = a^k$ (mod m), then k is called as the index of b modulo m relative to a. We write.

$$\text{Ind}_a\, b = k \text{ if } b \equiv a^k \pmod{m} \text{ or } \text{Ind}\, b = k.$$

Indices are like logarithms and their properties are also like logarithmic properties.

Example 2 : 2 is a primitive root mod 5 and $2^1 \equiv 2, 2^2 \equiv 4, 2^3 \equiv 3, 2^4 \equiv 1 \pmod 5$.

It follows that

$$\text{ind}_2\, 1 = 4, \text{ind}_2\, 2 = 1, \text{ind}_2 3 = 3, \text{ind}_2 4 = 2.$$

We observe that indices of integers which are congruent modulo n are equal.

If $a \equiv b$ (modn), where $(a,n) = (b,n) = 1$,

since $r^{\text{inda}} \equiv a \pmod n$ and $r^{\text{ind } b} \equiv b \pmod n$

we have $r^{\text{ind } a} \equiv r^{\text{ind } b} \pmod n$

THEOREM 14 : Let r be a primitive root of n. If $(a, n) = (b, n) = 1$ we have

(*a*) $\text{ind}\,(ab) = \text{ind}\, a + \text{ind}\, b \pmod{\varphi(n)}$

(*b*) $\text{ind}\, a^n = n\, \text{ind}\, a \pmod{\varphi(n)}$ if $n \geq 1$

(*c*) ind $1 = 0$ and ind $r = 1$. if $n > 2$.

PROOF : By definition of index,

$$r^{\text{ind } a} \equiv a \pmod n$$

and $r^{\text{ind } b} \equiv b \pmod n$. Multiplying these congruences together we get

$$r^{\text{ind } a + \text{ind } b} \equiv a\, b \pmod n$$

But $r^{\text{ind } (a\, b)} \equiv ab \pmod n$, so that

$$r^{\text{ind } a + \text{ind } b} \equiv r^{\text{ind}(\, a\, b)} \pmod n .$$

It may happen that ind a + ind b exceeds φ (n).

This gives no problem for by theorem the last equation holds if and only if the order are congruent modulo φ(n) ; i.e.

$$\text{ind } a + \text{ind } b \equiv \text{ind } (a\,b) \pmod{\varphi(n)}.$$

To prove (*b*).

$$r^{\text{ind } a^k} \equiv a^k \pmod n$$

while $\quad r^{k \text{ ind } a} \equiv (r^{\text{ind } a})^k \equiv a^k \pmod n$, hence

$$r^{\text{ind } a^k} \equiv r^{k \text{ ind } a} \pmod n$$

As in case (*a*) we have $\text{ind } a^k \equiv k \text{ ind } a \pmod{\varphi(n)}$.

The proof of (*c*), follow as:

$$r^0 \equiv 1 \pmod n\text{, we have ind } (1) = 0$$

$$a^1 \equiv a \pmod n \Rightarrow \text{ind}_a\,(a) = 1.$$

6.7. APPLICATION OF INDICES IN SOLVING CONGRUENCE.

The theory of indices can be used to solve certain types of congruence.

For example consider the binomial congruence

$$x^k \equiv a \pmod n$$

where n is a positive integer having a primitive root and gcd (a, n) = 1 By theorem 14 this congruence is equivalent to the linear congruence

$$k \text{ ind } x \equiv \text{ind } a \pmod{\varphi(n)}.$$

If $d = \gcd(k, \varphi(n))$ and $d \nmid \text{ind } a$, there is no solution . But if $d \mid \text{ind } a$, then there are exactly d values of ind x which will satisfy this last congruence, hence d incongruent solutions of $x^k \equiv a \pmod n$.

Consider the case k = 2 and n = p. We get the quadratic congruence

$$x^2 \equiv a \pmod p.$$

This congruence has a solution, if and only if $2 \mid \text{ind } a$; when this condition is fulfilled . there are exactly two solutions. If r is a primitive root of p, then r^k ($1 \le k \le p-1$) runs through integers 1, 2,... p–1, in some order . The even powers of r produces the values of a for which the congruence $x^2 \equiv a \pmod p$ is solvable. There are precisely (p–1)/2 such choices for a .

Example 1 : Solve the linear congruence $7x \equiv 2 \pmod 9$.

Solution : We know 2 is a primitive root modulo 9.

Also $\quad 2^1 \equiv 2 \pmod 9 \qquad 2^4 \equiv 7 \pmod 9$

$2^2 \equiv 4 \pmod 9 \qquad 2^5 \equiv 5 \pmod 9$

$2^3 \equiv 8 \pmod 9 \qquad 2^6 \equiv 1 \pmod 9$.

Index of 7 is 4 and 2 is 1.

Now $7x \equiv 2 \pmod{9}$ is equivalent to

$\text{Ind } 7 + \text{Ind } x \equiv \text{Ind } 2 \ (\text{mod } \varphi(9))$

$\equiv \text{Ind } 2 \pmod{6}$

or $4 + \text{Ind } x \equiv 1 \pmod{6}$

or $\text{Ind } x \equiv -3 \pmod{6}$.

Hence $x \equiv 2^3 \pmod{9}$ (Since $-3 \equiv 3$ mod (6))

or $x \equiv 8 \pmod{9}$.

Thus solutions of $7x \equiv 2 \pmod{9}$ are of the form $9t + 8$ for $t = 0, \pm 1, \pm 2, \ldots$

Example 2 : Solve the congruence

$$11x^3 \equiv 2 \pmod{23}.$$

Solution : Here $p = 23$, 5 is a primitive root of 23. Also

$5^1 \equiv 5 \pmod{23}$ $\quad$ $5^6 \equiv 8 \pmod{23}$

$5^2 \equiv 2 \pmod{26}$ $\quad$ $5^7 \equiv 17 \pmod{23}$

$5^3 \equiv 10 \pmod{26}$ $\quad$ $5^8 \equiv 16 \pmod{23}$

$5^4 \equiv 4 \pmod{23}$ $\quad$ $5^9 \equiv 11 \pmod{23}$

$5^5 \equiv 20 \pmod{23}$ $\quad$ $5^{10} \equiv 9 \pmod{23}$

$11x^3 \equiv 2 \pmod{23}$ is equivalent to

$\text{Ind } 11 + 3 \text{ Ind } x \equiv \text{Ind } 2 \pmod{22}$

Or $9 + 3 \text{ Ind } x \equiv \text{Ind } 2 \pmod{22}$

or $3 \text{ Ind } x \equiv -7 \pmod{22}$

or $3 \text{ Ind } x \equiv 15 \pmod{22}$

or $\text{Ind } x \equiv 5 \pmod{23}$

Hence $x \equiv 5^5 \pmod{23}$

or $x \equiv 20 \pmod{23}$.

Thus $x = 23t + 20$, $t = 0, \pm 1, \pm 2, \ldots$ are all solutions of the given congruence.

Exercise : Solve the congruence

$$4x^9 \equiv 7 \pmod{13}.$$

THEOREM 15 : Let n be an integer having a primitive root and let gcd $(a,n) = 1$, then the congruence $x^k \equiv a \pmod{n}$ has solution if and only if

(15) $$a^{\varphi(n)/d} \equiv 1 \text{ (mod n)},$$

where d = gcd (k, φ(n)); if it has a solution there are exactly d solutions modulo n.

PROOF : $a^{\varphi(n)/d} \equiv 1$ (mod n), implies φ(n)/d Ind a = Ind 1 (mod n) ≡ 0 (mod n) which holds if and only if d | ind a, which is a necessary and sufficient conditions for the congruence $x^k \equiv a$ (mod n) to be solvable.

Corollary : Let p be a prime and gcd (a,b) = 1. Then the congruence $x^k \equiv$ a (mod p) has a solution if, and only if $a^{p-1/d} \equiv 1$ (mod p), where d = gcd (k, p – 1).

Example 3 : Solve the congruence

$$x^3 \equiv 4 \text{ (mod 13)}.$$

Here, d = gcd (3, φ(13)) = g cd (3, 12) = 3.

So φ(13) / d = 4. Since $4^4 \equiv 9 \not\equiv 1$ (mod 13), the congruence is not solvable.

6.8. THE DISCRETE LOGARITHM

DEFINITON: Suppose r is a primitive root modulo n. If $r^x \equiv y$ (mod n) then the discrete logarithm or index of y (to the base r) is

$$\text{ind}_r(y) = x \text{ (mod } \varphi(n)).$$

Example 1 : 3 is a primitive root of 17. We have $3^8 \equiv -1$ (mod 17) and $3^{12} \equiv 4$ (mod 17).

So $\text{ind}_3(-1) = 8$ and $\text{ind}_3(4) = 12$.

Example 2 : We use the properties of discrete logarithm to solve the congruence

$$7^x \equiv 4 \text{ (mod 17)}.$$

Solution : 3 is a primitive root mod 17.

Taking the index to the base 3 we get

$$\text{ind}_3(7^x) \equiv \text{ind}_3(4)$$

or $$x\, \text{ind}_3(7) \equiv \text{ind}_3(4) \text{ (mod } \varphi(17)).$$

Since $\text{ind}_3(7) = 11$ and $\text{ind}_3(4) = 12$

solving the equation is equivalent to

$$11x \equiv 12 \text{ (mod 16)}.$$

The solution is $x \equiv 4$(mod 16).

EXERCISES

1. Find the index of 5 relative to each of the primitive roots of 13.

2. Using theory of indices find the remainder where 3^{24}. 5^{13} is divided by 17.

3. If r is a primitive root of n, verify that, $\text{ind}_r(-1) = \text{ind}_r(\varphi(n)) = \varphi(n)/2$.

4. Solve the congruence $x^3 \equiv 5 \pmod{13}$.

5. Solve the congruences

   (*i*) $7x^3 \equiv 3 \pmod{11}$.

   (*ii*) $4x \equiv 19 \pmod{23}$

   (*iii*) $57^x \equiv 2 \pmod{13}$.

6. Determine whether the congruence $x^5 \equiv 13 \pmod{23}$.is solvable.

7. For which values of b the congruence $9^x \equiv b \pmod{13}$ is solvable..

8. Using a table of indices for a primitive root of 11, solve the congruence

   (*i*) $7x^3 \equiv 3 \pmod{11}$.

   (*ii*) $3x^4 \equiv 5 \pmod{11}$

   (*iii*) $x^8 \equiv 10 \pmod{11}$.

9. If r and r' are both primitive roots of n, show that for gcd (a, n) = 1

   $\text{ind}_r a \equiv (\text{ind}_r a)(\text{ind}_{r'} r) \pmod{\varphi(n)}$.

10. Let p be an odd prime. Find all solutions of $x^{p-1} \equiv 2 \pmod{p}$.

11. Let a be a primitive root modulo prime p. Then prove that

    (*i*) $a^{(p-1)/2} \equiv -1 \pmod{p}$.

    (*ii*) If a′ is another primitive root modulo p, then aa′ is not a primitive root modulo p.

12. Let p be an odd prime. Prove that a has order 2 mod p if and only if $a \equiv -1 \pmod{p}$.

13. Prove that n is a prime if and only if $\text{ord}_n(a) = n - 1$ for some a.

14. If (a, n) = (b,n) = 1 and if $(\text{ord}_n(a), \text{ord}_n(b)) = 1$, prove that $\text{ord}_n(ab) = \text{ord}_n(a)\text{ord}_n(b)$.

15. Show that 18 is a primitive root modulo 37. Is it also a primitive root modulo 37^2 ?

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CHAPTER

7

QUADRATIC RESIDUES AND QUARATIC RECIPROCITY LAWS

7.1. QUADRATIC RESIDUES

In chapter 3, the problem of solving polynomial congruence

$$f(x) \equiv 0 \pmod{p}$$

are dealt with . Here we will consider quadratic congruence of the form

(1) $$x^2 \equiv a \pmod{p}$$

where p is an odd prime and $(a,p) = 1$.

Lagranges theorem tells that a polynomial congruence

$$f(x) = a_0 + a_1 x + \ldots + a_n x^{-n} \equiv 0 \pmod{p},$$

$$(a_n, p) = 1 \text{ has atmost n solutions.}$$

So the quadratic congruence (1) has atmost two solutions. If x is a solution so is $-x$, hence (1) has either no solution or 2 solutions.

Now consider the general quadratic congruence

(2) $$ax^2 + bx + c \equiv 0 \pmod{p}$$

where p is an odd prime and $(a,p) = 1$. p is an odd prime implies that $(4a,p) = 1$. Then (2) is equivalent to

(3) $$4a(ax^2 + bx + c) \equiv 0 \pmod{p}.$$

But $4a(ax^2 + bx + c) = (2ax + b)^2 - (b^2 - 4ac)$.

So (3) may be expressed as

(4) $$(2ax + b)^2 \equiv (b^2 - 4ac) \pmod{p}.$$

Now put $y = 2ax + b$ and $d = b^2 - 4ac$ to get

(5) $$y^2 \equiv d \pmod{p}.$$

If $x \equiv x_0 \pmod{p}$ is a solution of (2) then

$$y = 2ax_0 + b \pmod{p}$$

satisfies the congruence (5). Conversely if $y \equiv y_o \pmod{p}$ is a solution of (5), then

$$2ax \equiv y_o - b \pmod{p}$$

can be solved to obtain solution of (2).

Thus, the problem of finding a solution to the quadratic congruence (2) is equivalent to that of finding a solution to linear congruence and a quadratic congruence of the form (1).

It is clear that if $x = x_0$ is a solution of (1) and $x = p-x_0$ is another solution.

Example 1 : Solve $x^2 \equiv 5 \pmod{29}$.

Solution :

$$x^2 \equiv 5 \equiv 5+29 \equiv 34 \equiv 63 \equiv 92 \equiv 121 \equiv 11^2$$

(mod p) (i.e. add multiple of p till we get a square number).

Hence the required solutions are

$$x \equiv 11, 29-11 \pmod{29}$$
$$\equiv 11, 18 \pmod{29}.$$

Example 2 : Solve $x^2 \equiv 4 \pmod{13}$.

Solution :

$x^2 \equiv 4 \equiv 17 \equiv 30 \equiv 43 \equiv 56 \equiv 69 \equiv 82 \equiv 95 \equiv 108 \equiv 121 \equiv 11^2$ (mod 13)

So solution are $x \equiv 11, 13\text{-}11 \pmod{29}$

$$\equiv 11, 2 \pmod{29}.$$

Example 3 : Solve the congruence

$$2x^2 - 5x + 3 \equiv 0 \pmod{29}.$$

Solution : The congruence is reduced to $y^2 \equiv d \pmod{29}$

where $d = b^2 - 4ac = 25 - 24 = 1$

and to solve the congruence $y^2 \equiv 1 \pmod{29}$

with solution $y \equiv 1, 28 \pmod{29}$.

Next solve the linear congruence

$$4x \equiv 1-(-5) \equiv 6 \pmod{29}$$
$$4x \equiv 28-(-5) = 33 \equiv 4 \pmod{29}$$
$$x \equiv 6.\,4^{28-1} \pmod{29} \equiv 6.4^{27} \pmod{29}.$$

To calculate 4^{27} (mod 29). $4^{28} \equiv 1 \pmod{29}$.

$\Rightarrow$ $4^{27} \equiv 4^{-1} \pmod{29}$.

To calculate 4^{-1} (mod 29) .

$$4.4^{-1} \equiv 1 \pmod{29}$$

$\Rightarrow$ $\qquad 4^{-1} = 22.$

Hence $\qquad x \equiv 6.22 \pmod{29} \equiv 16 \pmod{29}$

$\Rightarrow x \equiv 16 \pmod{29}$ is a solution which satisfies the original quadratic congruence also .

Next $\qquad 4x \equiv 4 \pmod{29}$

$\Rightarrow$ $\qquad x \equiv 1 \pmod{29}$.

So $x \equiv 1 \pmod{29}$ is a solutions of the congruence $4x \equiv 4 \pmod{29}$ hence of the original quadratic congruence.

Hence the two solution are $x \equiv 1, 16 \pmod{29}$.

Exercise 1 : Solve $5x^2 - 6x + 2 \equiv 0 \pmod{13}$.

Exercise 2 : Solve $x^2 \equiv 52 \pmod{101}$.

DEFINITION : If the congruence $x^2 \equiv a \pmod{p}$, where p is a prime and $(a,p) = 1$ has a solution then a is said to be a quadratic residue modulo p. We write aRp Otherwise a is called quadratic nonresidue modulo p. We write aNp.

If $a \equiv b \pmod{p}$ then $(a,p) = (b,p)$ so a is a quadratic residue modulo p if and only if b is a quadratic residue modulo p.

Example : Consider $p = 17$. Find all quadratic residues and non residues of 17.

Solution : We find a among [1,2,3,.....1 6] modulo 17 which satisfy the congruence

$$x^2 \equiv a \pmod{17}.$$

The squares of the integer 1, 2, 3,....16 are

$$1^2 \equiv 16^2 \equiv 1$$

$$2^2 \equiv 15^2 \equiv 4$$

$$3^2 \equiv 14^2 \equiv 9$$

$$4^2 \equiv 13^2 \equiv 16$$

$$5^2 \equiv 12^2 \equiv 8$$

$$6^2 \equiv 11^2 \equiv 2$$

$$7^2 \equiv 10^2 \equiv 15$$

$$8^2 \equiv 9^2 \equiv 13.$$

So the quadratic residues of 17 are 1, 2,4, 8, 13, 15, 16 while non residues are 3,5,6,7, 10, 11, 12, 14.

Remark : The number of quadratic residue of p are same as the number of quadratic non residues.

Example : If $p = 19$ then $(p-1)/ 2 = (19-1)/2 = 9.$

Hence $x^2 \equiv 1 \pmod{19}$ has exactly 9 solutions, namely

$$x \equiv 1^2, 2^2, \ldots\ldots 9^2 \pmod{19}$$

$$\equiv 1, 4, 9, 16, 6, 17, 11, 7, 5. \pmod{19}.$$

THEOREM 1 : If $(a,p) = 1$ then $a^{(p-1)/2} \equiv \pm 1 \pmod{p}$.

PROOF : By Fermat's theorem $a^{p-1} \equiv 1 \pmod{p}$.

This $\Rightarrow (a^{(p-1)/2} - 1) (a^{(p-1)/2} +1) \equiv 0 \pmod{p}$

$\Rightarrow$ either $a^{(p-1)/2} -1 \equiv 0 \pmod{p}$ i.e. $a^{(p-1)/2} \equiv 1 \pmod{p}$

or $a^{(p-1)/2} +1 \equiv 0 \pmod{p}$ i.e. $a^{(p-1)/2} \equiv -1 \pmod{p}$.

But not both since in that case $(a^{(p-1)/2} +1) - (a^{(p-1)/2} -1) = 2$ would be divisible by p which is impossible.

Hence the theorem is proved.

THEOREM 2 : The congruence

(6) $x^{(p-1)/2} \equiv 1 \pmod{p}$

has exactly p-1/2 solutions congruent module p and p-1/2 solutions incongruent modulo p. The congruent solutions are

$$x \equiv 1^2, 2^2, \ldots (p-1/2)^2 \pmod{p}.$$

PROOF : Let $S = \{ 1^2, 2^2, .. (p-1/2)^2 \}$.

If t^2 is an integer of S, then $(t,p) = 1$.

So by Fermat's theorem we have

$$t^{p-1} \equiv 1 \pmod{p}$$

which $\Rightarrow$ $(t^2)^{p-1/2} \equiv 1 \pmod{p}$.

Thus every integer of S is a solution.

All the integers of S are all incongruent (mod p).

For if $u^2 \equiv v^2 \pmod{p}$ such that $1 \leq u \leq v \leq (p-1)/2$ then it would follow that $(u - v) (u + v)$ would be divisible by p. But this is impossible since both $u - v$ and $u + v$ are numerically less than p. Moreover $(p-1)/2$ divides $p-1$. Therefore (6) has exactly $(p-1)/2$ solutions.

Since there are p-1/2 solutions of the congruence $x^2 \equiv a \pmod{p}$, there are exactly p-1/2 incongruent solutions mod p. Hence we have the following .

THEOREM 3 : Let p be an odd prime. Then every reduced residue system mod p contains exactly (p-1)/2 quadratic residues and exactly (p-1)/2 quadratic non-residues mod p. The quadratic residues belong to the residue classes containing the numbers.

(7) $$1^2, 2^2, 3^2, \dots ((p-1)/2)^2.$$

7.2. LEGENDRE'S SYMBOL AND ITS PROPERTIES.

DEFINITION : Let p be a prime. We define Legendre's symbol (a/p) as follows:

(8) $$\left(\frac{a}{p}\right) = \begin{cases} +1 \text{ if } aRp \\ -1 \text{ if } aNp. \end{cases}$$

$$(a/p) = 0 \text{ if } p|a.$$

Example : (1/p)= 1, p is a prime.

$(a^2/p) = 1$ (8/17) = 1 (7/19) = 1.

(7/11) = 1, since 7 is a quadratic residue modulo 11.

THEOREM 4 : (Euler's criterion). The congruence

$$x^2 \equiv a \pmod{p}, (a,p) = 1$$

has a solution if and only if

$$a^{(p-1)/2} \equiv 1 \pmod{p}.$$

PROOF : Let

$$a^{(p-1)/2} \equiv 1 \pmod{p}.$$

Then a is solution of

$$x^{(p-1)/2} \equiv 1 \pmod{p}.$$

Therefore a is congruent mod p to one of the integers $1^2, 2^2, \dots ((p-1)/2)^2$. Let this integer be t^2. That is $t^2 \equiv a \pmod{p}$. Therefore x = t is a solution of $x^2 \equiv a (\text{mod } p)$.

Converse part : Let the quadratic congruence $x^2 \equiv a \pmod{p}$ has a solution say

$$x \equiv b \pmod{p}. \text{ It follows that } b^2 \equiv a \pmod{p}.$$

Hence $a^{p-1/2} \equiv (b^2)^{p-1/2} \equiv b^{p-1} \pmod{p} \equiv 1 \pmod{p}$ by Fermat's theorem.

Example : Show that $x^2 = 18 \pmod{79}$ is soluble.

Solution : Herep = 79, (p-1)/2 = 39, a = 18, $a^{(p-1)/2} \equiv 18^{39} \equiv 1 \pmod{79}$.

Hence the given congruence is solvable . The solution can be found as in previous example They are x ≡ 27, 52 (mod 79).

THEOREM 5 : $x^2 \equiv a \pmod p$, $(a,p) = 1$ has no solution if and only if

$$a^{(p-1)/2} \equiv -1 \pmod p.$$

PROOF : Suppose $a^{(p-1)/2} \equiv -1 \pmod p$.

Then it follow $a^{(p-1)/2} \not\equiv 1 \pmod p$.

Therefore $x^2 \equiv a \pmod p$ has no solution.

Conversely suppose $x^2 \equiv a \pmod p$ be not solvable.

Then it follow that $a^{(p-1)/2} \equiv 1 \pmod p$.

So $a^{(p-1)/2} \equiv -1 \pmod p$.

Example 1. $x^2 \equiv 10 \pmod{19}$ has no solution since $10^{(19-1)/2} \equiv 10^9$

$\equiv -1 \pmod{19}$.

Example 2. $x^2 \equiv 10 \pmod{11}$ has no solution since $10^{(11-1)/2} = 10^5$

$\equiv -1 \pmod{11}$.

THEOREM 6 : (Euler' criterion) $x^2 \equiv a \pmod p$, $(a,p) = 1$ is solvable if and only if

$a^{(p-1)/2} \equiv 1 \pmod p$ and has no solution if and only if

$a^{(p-1)/2} \equiv -1 \pmod p$.

Corollary : (Fermat's theorem). From the above theorem we deduce Fermat's theorem. Because there are two possibilities. The congruence $x^2 \equiv a \pmod p$ has a solution or has no solution .

We have either

$$a^{(p-1)/2} \equiv 1 \pmod p \text{ or}$$

$$a^{(p-1)/2} \equiv (-1) \pmod p.$$

Squaring both the sides we obtain

$$a^{p-1} \equiv 1 \pmod p.$$

Since $(a/p) = +1$ if $a R p$

$= -1$ if $a N p$

in Euler's Criterion we replace the right hand side by (a/p) and we have,

RESULT : Let p be an odd prime then for all a we have

$$(a/p) \equiv a^{(p-1)/2} (\mathrm{mod} p).$$

SOME PROPERTIES OF LEGENDRE'S SYMBOL

THEOREM 7 : Legendre's symbol (a/p) is a completely multiplicative function.

PROOF : To prove that (ab/p) = (a/p) (b/p) for all a and b.

(i) If p|a or p|b then p|ab. Now p|a $\Rightarrow$ (a/p) =0 or p| b $\Rightarrow$ (b/p) = 0.

and p|ab $\Rightarrow$ (ab/p) = 0

Hence (ab/p) = (a/p) (b/p).

(ii) If p$\nmid$a and p$\nmid$ b then p| ab and we have

$$(ab/p) \equiv (ab)^{(p-1)/2} \equiv a^{(p-1)/2}\, b^{(p-1)/2} \equiv (a/p)\,(b/p) \pmod p$$

by theorem 6 But each of (ab/p), (a/p) and (b/p) is 1 or –1 so the difference (ab/p) – (a/p)(b/p) is either 0, 2 or –2.

THEOREM 8 : If a $\equiv$ b (mod p). Then (a/p) = (b/p).

PROOF : If a is a quadratic residue modulo p so is b and if a quadratic non residue modulo p then so is b.

THEOREM 9 : $(a_1/p)\,(a_2/p) \ldots. (a_k/p) = (a_1a_2..a_k/p)$.

PROOF : $(a_1/p)\,(a_2/p)\ldots\,(a_k/p) \equiv a_1^{p-1/2} \ldots\ldots a_k^{p-1/2} \pmod p$

$$\equiv (a_1\ldots.a_k)^{p-1/2} \pmod p$$

$$\equiv (a_1\ldots\ldots a_k/p) \pmod p$$

Since $(a_1/p), (a_2/p) \ldots (a_k/p)$ are all equal to + 1 or –1.

Similarly $(a_1..a_k/p) = +1$ or -1.

Hence $(a_1/p)\,(a_2/p) \ldots (a_k/p) = (a_1\, a_2 \ldots a_k/p)$.

THEOREM 10 : If (a,p) = 1, then $(a^2/p) = 1$

PROOF : The theorem is trivial since a^2 is a quadratic residue of every p.

Corollary : $(a\,b^2/p) = (a/p)$.

PROOF : $(a\,b^2/p) = (a/p)\,(b^2/p) = (a/p)$.

THEOREM 11 : For every odd prime p we have

$$(9) \qquad (-1/p) = (-1)^{p-1/2} = \begin{cases} 1 \text{ if } p = 4k+1 \\ -1 \text{ if } p = 4k+3 \end{cases}$$

PROOF : By Euler's Criterion we have $(-1/p) = (-1)^{p-1/2} \pmod p$

Since the values of both sides of congruence are 1 or –1 the two members are equal.

That is

$$(-1/p) = (-1)^{p-1/2}$$

If $p = 4k + 1, (-1)^{p-1/2} = (-1)^{2k} = +1$

If $p = 4k + 3, (-1)^{p-1/2} = (-1)^{2k+1} = -1$

Hence proved.

THEOREM 12 : For every odd prime p we have

$$(10) \qquad (2/p) = (-1)^{\frac{p^2-1}{8}} \begin{cases} 1 \text{if } p \equiv \pm 1 (\text{mod } 8) \\ -1 \text{if } p \equiv \pm 3 (\text{mol } 8). \end{cases}$$

PROOF : we know

$$(A) \qquad \begin{cases} (p-1) \equiv 1(-1)^1 \ (\text{mod } p) \\ 2 \equiv 2(-1)^2 (\text{mod } p) \\ p-3 \equiv 3(-1)^3 (\text{mod } p) \\ 4 = 4(-1)^4 (\text{mod} p) \\ \\ r \equiv (p-1)/2(-1)^{(p-1)/2} (\text{mod } p), \end{cases}$$

where r is either p-1/2 or p- (p-1/2) multiplying together in the set *(A)* we obtain

$$2, 4, 6..... (p-1) \equiv (p-1/2)\,!\,(-1)^{1+2+...+(p-1)/2} \ (\text{mod } p)$$

$$\Rightarrow \qquad 2^{(p-1)/2} (p-1/2)! \equiv (p-1/2)\,!\,(-1)^{p2-{}_{1/8}} \ (\text{mod } p).$$

Since $(p, (p-1/2)\,!\,) = 1$ canceling $((p-1)/2)\,!$

from both sides we obtain

$$2^{(p-1)/2} \equiv (-1)^{(p2-1)/8} \ (\text{mod } p).$$

But by Euler's criterion $2^{(p-1)/2} \equiv (2/p) \ (\text{mod } p)$.

and since each member is + 1 or –1 the two members are equal completing the proof of the theorem.

7.3. GAUSS' LEMMA

We now give another Criterion due to Gauss which involves a simpler calculation and applicable for large n.

THEOREM 13 : (Gauss'lemma) Assume (a, p) = 1 and consider the least positive residues modulo p of the following (p-1/2) multiples of a.

(11) a, 2a, 3a, ..., (p-1/2) a.

If n denotes the number of these residues which exceed p/2, then

$$(a/p) = (-1)^n.$$

PROOF : Let $r_1, r_2, \dots r_n$ denote the residues that exceed p/2, and let $s_1, s_2 \dots s_k$ denote the remaining residues. The r_i and s_i are all distinct, and non zero.

Further $n + k = (p-1)/2$. Now $0 < p-r_i < p/2$, $i = 1, 2, \dots n$, and the numbers $p-r_i$ are distinct. Also no $p-r_i$ is an s_j for if $p-r_i = s_j$ then $r_i \equiv \rho a$, $s_j \equiv \sigma a$, for some ρ, σ, $1 \le \rho \le p-1/2$, $1 \le \sigma \le (p-1)/2$, and $p - \rho a \equiv \sigma a \pmod p$. Since $(a,p) = 1$ this implies $a(\rho+\sigma) \equiv 0$, $(\rho+\sigma) \equiv 0 \pmod p$ which is impossible by the nature of ρ and σ. Thus $p-r_1, p-r_2 \dots, p-r_n, s_1, s_2 \dots s_k$ are all distinct, are all at least 1 and less than p/2.

They are just the integer 1,2, ... (p-1)/2 in some order. Multiplying them together we have

$$(p-r_1)(p-r_2)\dots(p-r_n)\, s_1 s_2 \dots s_k$$
$$= 1.2.3 \dots\dots p-1/2.$$

and then

$$(-r_1)(-r_2)\dots(-r_n)\, s_1 s_2 .. s_k \equiv 1,2.. p-1/2 \pmod p$$
$$\Rightarrow (-1)^n r_1 r_2 \dots r_n s_1 s_2 \dots s_k \equiv 1.2\dots p-1/2 \pmod p$$
$$\Rightarrow (-1)^n 2a \dots p-1/2\, a \equiv 1.2.3\dots p-1/2 \pmod p$$
$$\Rightarrow (-1)^n a^{p-1/2} (p-1)/2\,! \equiv (p-1/2)\,! \pmod p$$

Since ((p-1/2)!, p)= 1, cancelling (p-1)/2 ! from both sides we have

$$(-1)^n a^{p-1/2} \equiv 1 \pmod p$$

$\Rightarrow (-1)^n = a^{p-1/2} \equiv (a/p) \pmod p$ by Euler's criterion.

Hence $$(a/p) = (-1)^n.$$

THEOREM 14 : Let n denotes the number of those residues which exceed p/2 then

$$n = \sum_{t=1}^{(p-1)/2} [ta/p] + (a-1)^{(p2-1)/8} \pmod 2.$$

In particular, if n is odd we have $n \equiv \sum_{t=1}^{(p-1)/2} [ta/p] \pmod 2$.

PROOF : Consider the numbers

a, 2a, 3a.... (p-1/2) a.

Take ta and divide it by p then

$$ta/p = [ta/p] + \{ta/p\}, \text{ where } 0 < \{ta/p\} < 1.$$

So ta≡ p [ta/p] + p {ta/p} = p [ta/p] + t_i , for some t_i

where $0 < t_i < p$.

t_i = ta – p [ta/p] is the least positive residue of ta modulo p. Now

$$\{t_1, t_2, \ldots t\,(p\text{-}1/2)\} = \{ r_1, r_2 \ldots r_n, s_1, s_2 \ldots s_k\}$$

and $$\{1, 2 \ldots p\text{-}1/2\} = \{r_1, r_2 \ldots r_n , s_1 \ldots s_k\}.$$

So since (a, p) = 1, whether a is odd or even, we have

$$\sum_{t=1}^{(p-1)/2} ta \equiv \sum_{t=1}^{p-1/2} p\,[ta/p] + \sum_{i=1}^{p-1/2} t_i$$

$$\equiv \sum_{t=1}^{p-1/2} p\,[ta/p] + \sum_{i=1}^{n} r_i + \sum_{j=}^{k} s_j$$

and $$\sum_{t=1}^{p-1/2} t = \sum_{i=1}^{n} (p\text{-}r_i) + \sum_{j=}^{k} s_j$$

$$= np - \sum_{i=1}^{n} r_t + \sum_{j=}^{k} s_j$$

and hence by subtraction,

$$(a\text{-}1) \sum_{t=1}^{(p-1/2)} t = p\,\Big(\sum_{t=1}^{p-1/2} [ta/p] - n \Big) + 2\sum_{i=1}^{n} r_i.$$

But

$$\sum_{t=1}^{(p-1/2)} t = 1 + 2 + \ldots (p\text{-}1/2) = p^2\text{-}1/8.$$

So we have

$$(a\text{-}1)^{(p2-1)/8} \equiv \sum_{t=1}^{(p-1/2)} [ta/p] - n \pmod 2.$$

If a is odd, this implies

$$n \equiv \sum_{t=1}^{(p-1/2)} [ta/p] \pmod 2.$$

If a = 2, it implies $n \equiv p^2\text{-}1/8$.

Since

$$[2t/p] = 0 \text{ for } 0 \leq t \leq p\text{-}1/2$$

we get

$$(2/p) = (-1)^{p^2-1/8}$$

which is the theorem 12.

7.4. QUADRATIC RECIPROCITY LAW AND ITS APPLICATION.

THEOREM 15 : If p and q are distinct odd primes, then

(12) $(p/q)\ (q/p) = (-1)^{(p-1)\ (q-1)/4}$.

PROOF : By Gauss Lemma (theorem 13) and theorem 14 we have

$$(q/p) = (-1)^n$$

where

$$n \equiv \sum_{t=1}^{(p-1/2)} [tq/p] \pmod 2.$$

Similarly $(p/q) = (-1)^m$,

where

$$m \equiv \sum_{s=1}^{(q-1/2)} [sp/q] \pmod 2.$$

Hence $(p/q)\ (q/p) = (-1)^{m+n}$

where $m + n = \displaystyle\sum_{t=1}^{(p-1/2)} [tq/p] + \sum_{s=1}^{(q-1/2)} [sp/q]$.

Let s be the set of all pairs of integers (x,y) satisfying

$1 \leq x \leq (p\text{-}1)/2,\ 1 \leq y \leq (q\text{-}1)/2.$

The set has (p-1) (q – 1) /4 members. Separate the set into two mutually exclusive subsets S_1 and S_2 according as qx > py or qx <py. There are no pairs (x,y) in S such that qx = by S_1 is the set of all pairs (x,y) such that

$1 \leq x \leq (p\text{-}1)/2,$

$1 < y < qx/p.$

The number of pairs in S_1 is seen to be $\displaystyle\sum_{x=1}^{(p-1)/2} [qx/p]$. Similarly the number of pairs (x,y) in S_2 s.t $1 \leq y \leq q-1/2$,

$1 \leq x \leq py/q$ is

$\sum_{y=1}^{(q-1)/2} [py/q]$. Thus we have m + n = (p-1)/2 (q-1)/2.

Hence

$$\cdot\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}.$$

THEOREM 16 : If 8k + 7 is a prime then it divides M_{4k+3} .

PROOF : We have proved in theorem 12 that 2 is a quadratic residue of prime of the form 8k-1, i.e. of the form 8k+7. Hence by Euler's criterion

$2^{8k+7-1/2} \equiv 1 \pmod{8k+7}$.

This implies

$$2^{4k+3} \equiv 1 \pmod{8k+7}$$

$$\Rightarrow 2^{4k+3} - 1 \equiv 0 \pmod{8k+7}$$

i.e. $8k + 7 \mid M_{4k+3}$

This proves the theorem.

Example : Apply the above theorem to find a factor each of some of the Mersenne numbers M_p where p is a prime ≤ 257.

Solution : Since by the theorem 8k + 7 divides M_{4k+3}

k	8k + 7	4k + 3	Factor of M_{4k+3}
2	23	11	Hence $23 \mid M_{11}$
5	47	23	$47 \mid M_{23}$
20	167	83	$167 \mid M_{83}$
32	263	131	$263 \mid M_{131}$
44	359	179	$359 \mid M_{179}$
47	383	191	$383 \mid M_{191}$
59	479	239	$479 \mid M_{239}$
62	503	251	$503 \mid M_{251}$.

THEOREM 17 : - 2 is a quadratic residue of p if and only if p is of the form 8k+1, or 8k+3

PROOF : (-2/p) = (-1/p) (2/p).

$$\begin{aligned}(-2/p) &= (1)\,(1) = 1 \text{ if } p = 8k + 1\\ &= (-1)\,(-1) = 1 \text{ if } p = 8k + 3\\ &= (1)\,(-1) = -1 \text{ if } p = 8k + 5\\ &= (-1)\,(1) = -1 \text{ if } p = 8k + 7.\end{aligned}$$

The theorem is therefore proved.

THEOREM 18 : 3 is a quadratic residue of p if and only if p is of the form 12 k ± 1.

PROOF : Every prime p > 3 has one of the forms 12k + 1, 12k – 1, 12k +5 and 12k – 5. Let p = 2k + 1.

Then

$$\left(\frac{3}{p}\right) = \left(\frac{12k+1}{3}\right) = \left(\frac{1}{3}\right) = 1.$$

If p = 12k –1 = 4 (3k –1) + 3 then

$$\left(\frac{3}{p}\right) = \left(\frac{12k+1}{3}\right) = (1)\left(\frac{-1}{3}\right) = 1.$$

If p = 12k + 5 = 4 (3k+1)+1. Then

$$\left(\frac{3}{p}\right) = \left(\frac{12k+5}{3}\right) = \left(\frac{-1}{3}\right) = 1.$$

If p = 12k - 5 = 4 (3k-2) +3 . Then

$$\left(\frac{3}{p}\right) = \left(\frac{12k-5}{p}\right) = -\left(\frac{1}{3}\right) = 1.$$

This proves the theorem.

Example : -3 is a quadratic residue of p if and only if p is of the form 6k + 1 and quadratic non-residue if and only if p is of the form 6k + 5.

Solution : p is odd , hence

(13) $$p \equiv 1 \pmod 2$$

$$\begin{aligned}(-3/p) &= (-1/p)\,(3/p)\\ &= (p/3) \text{ if } p = 4k + 1\end{aligned}$$

$$(-3/p) = (-1/p)\,(3/p) = (p/3) \text{ if } p = 4k + 3$$

It follows that (-3/p) = 1 if and only if p is a quadratic residue of 3. This implies

(14) $$p \equiv 1 \pmod{3}.$$

From (13) and (14) we get $p \equiv 1$ (mod 6) i.e,

$$p = 6k + 1$$

-3 is a quadratic non-residue of a prime if and only if p is the form

6k + 5 or 6k + 1.

Example : 5 is a quadratic residue of a prime p if and only if p is of the form 10k ± 1 .

Solution : Every odd prime p has one of the forms 10k ± 1 and 10k ± 3. Let p = 10k ± 1.

Then p = 10k + 1. Then

$$\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{10k+1}{5}\right) = +\left(\frac{1}{5}\right) = 1.$$

Let p = 10k – 1. Then

$$\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{10k-1}{5}\right) = \left(\frac{-1}{5}\right) = 1.$$

Let p = 10k +3 . Then

$$\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{10k+3}{5}\right) - \left(\frac{3}{5}\right) = \left(\frac{5}{3}\right) \quad = \left(\frac{-1}{3}\right) = -1.$$

Let p = 10k – 3 . Then

$$\left(\frac{5}{p}\right) = \left(\frac{p}{5}\right) = \left(\frac{-3}{5}\right) = \left(\frac{2}{5}\right) = (1)^{\frac{5^2-1}{8}} = (-1)^3 = -1.$$

Exercise 1 : Prove that 5 is a quadratic non-residue of a prime p if and only if p is of the form 10k + 3.

Exercise 2 : Prove that –5 is a quadratic residue p if and only if p is of the form 20k + 1, 20k + 3, 20k + 7 or 20k + 9.

Exercise 3 : Prove that 7 is a quadratic residue of p if and only if p is of the form 28k ± 1, 28k ± 3 or 28k ± 9.

Exercise 4 : Prove that 10 is a quadratic residue of p if and only if p is of the form 40k ± 1, 40k ± 3 40k ± 9 or 40k ± 13.

Example : Find all quadratic residues of 35.

Solution : 35 = 5x7 and (5,7) = 1. Therefore the quadratic residues of 35 are those integers, less than 35, which are quadratic residues of both 5 and 7 . The quadratic residues of 5 which are less than 35 are

1, 4, 6, 9, 11, 14, 16, 19, 21, 24, 26, 29, 31, and 34.

The quadratic residues of 7 which are less than 35 are

1, 2, 4, 8, 9, 11, 15, 16, 18, 22, 23, 25, 29, 30, and 32.

Hence the quadratic residues of 35 are those integers which are common to both the above lists, namely

1, 4, 9, 11, 16 and 29.

Exercise : Find the quadratic residues of 25.

Solution : 25 $= 5^2$. Every quadratic residue of 5 is a quadratic residue of 25 and vice versa . Thus the least quadratic residues of 25 are those quadratic residues of 5 which are less than 5. The least quadratic residuue of 5 are 1 and 4. Hence quadratic residues of 25 are 1, 4, 6, 9, 11, 14, 16, 19, 21, and 24.

Exercise : Find all quadratic residue of 50.

Example : Evaluate

$$\left(\frac{-42}{61}\right).$$

Solution :

$$\left(\frac{-42}{61}\right)=\left(\frac{-1}{61}\right)\left(\frac{3}{61}\right)\left(\frac{3}{61}\right)\left(\frac{7}{61}\right)$$

$$\left(\frac{-1}{61}\right)=(-1)^{61-1/2}=1.$$

$$\left(\frac{2}{61}\right)=(-1)^{61-1/8}=-1.$$

Again by Reciprocity law .

$$\left(\frac{3}{61}\right)=\left(\frac{61}{3}\right)(-1)^{(2/2)(60/2)}=\left(\frac{1}{3}\right)=1.$$

$$\left(\frac{7}{61}\right)\cdot\left(\frac{61}{7}\right)(-1)^{(2/2)(60/2)}=\left(\frac{5}{7}\right).$$

$$\left(\frac{7}{5}\right)=(-1)^{(4/2)(6/2)}=\left(\frac{2}{5}\right)$$

Hence $\left(\frac{-42}{61}\right)=1.$

2nd Method Also we can evaluate $\left(\frac{-42}{61}\right)$ as follows :

Since –42 ≡ 19 (mod 61)

$$\left(\frac{-42}{61}\right)=\left(\frac{19}{61}\right)=\left(\frac{61}{19}\right)=(-1)^{(19-1)(61-1)/4}=\left(\frac{61}{19}\right)$$

$$=\left(\frac{4}{19}\right)= \ .$$

Since 61 ≡ 4 (mod 19) .

7.5. THE JACOBI SYMBOL

Jacobi extended Legendre's symbol (a/p) for composite number and formed analogous results for the new Jacobi symbol. We define it as follows:

DEFINITION : Let (P,Q) = 1, Q > 0, Q odd, so that $Q = q_1, q_2 \ldots q_s$ where the q_i's are not necessarily distinct. The Jacobi symbol (P/Q) is de fined by

$$\left(\frac{P}{Q}\right)=\prod_{i=1}^{s}\left(\frac{P}{q_i}\right) \tag{15}$$

where $\left(\frac{P}{q_i}\right)$ is Legendre symbol.

Remark : If Q is an odd prime then Jacobi symbol is same as Legendre symbol.

Clearly (P/Q) = ± 1. But it is not true that

(P/Q) = 1

⇒ P is a quadratic residue modulo Q .

For example (2/9) = 1 but $x^2 \equiv 2 \pmod 9$ has no solution.

THEOREM 19 : Suppose that Q and Q′ are odd and positive and that (PP′, QQ′) = 1, then

(i) $\left(\frac{P}{Q}\right)\left(\frac{P}{Q'}\right) = \left(\frac{P}{QQ'}\right)$

(ii) $\left(\frac{P}{Q}\right)\left(\frac{P'}{Q}\right) = \left(\frac{PP'}{Q}\right)$

(iii) $\left(\frac{P^2}{Q}\right) = \left(\frac{P}{Q^2}\right) =$

(iv) $\left(\frac{P' P^2}{Q' Q^2}\right) = \left(\frac{P'}{Q'}\right)$

(v) $P \equiv P \pmod{Q} \Rightarrow \left(\frac{P'}{Q}\right)\left(\frac{P}{Q}\right)$

PROOF : *(i)* follows from the definition of (P/Q) and

(ii) follows from the definition and from the properties of Legendre's symbol.

(iii) follows from *(i)* and *(ii)* so also *(iv)*. For *(v)* we write $Q = q_1 q_2 \ldots q_s$.

Then $P' \equiv P \pmod{q_j}$ so that $(P'/q_j) \equiv (P/q_j)$ Hence *(v)* follows from the definition.

THEOREM 20 : If Q is odd and Q > 0, then

$$(16)\quad \left(\frac{-1}{Q}\right) = (-1)^{Q-1/2} \text{ and } \left(\frac{2}{Q}\right) = (-1)^{(Q^2-1)/8}.$$

PROOF : We have

$$\left(\frac{-1}{Q}\right) = \prod_{j=i}^{s}\left(\frac{-1}{q_j}\right) = \prod_{j=i}^{s}(-1)^{(q_j-1)/2}$$

$$= (-1)^{\sum_{j=1}^{s}(q_j-1)/2}.$$

If a and b are odd, then

$$\frac{ab - 1}{2} - \left(\frac{a-1}{2} + \frac{b-1}{2}\right) = \frac{(a-1)(b-1)}{2} \equiv 0 \pmod{2}$$

and hence

$$\frac{a-1}{2}+\frac{b-1}{2}\equiv\frac{ab-1}{2} \pmod{2}.$$

Applying this repeatedly we obtain

$$\sum_{j=1}^{s}\left(\frac{q_1-1}{2}\right)=\frac{1}{2}\sum_{j=1}^{s} q_j \quad 1\equiv\left(\frac{Q-1}{2}\right) \pmod{2}.$$

Thus

$$\left(\frac{-1}{Q}\right)=(-1)^{Q-1/2}.$$

Similarly if a and b are odd, then

$$\frac{a^2b^2-1}{8}\left(\frac{a^2-1}{8}+\frac{b^2-1}{8}\right)=\frac{(a^2-1)\,(b^2 \quad 1)}{8}\equiv 0 \pmod{8}.$$

So we have

$$\frac{a^2-1}{8}+\frac{b^2-1}{8}\equiv\frac{(a^2b^2-1)}{8} \pmod{2}$$

$$\sum_{j=1}^{s}\frac{q_j^2-1}{8}=\frac{Q^2-1}{8} \pmod{2}.$$

and hence

$$\left(\frac{2}{Q}\right)=\prod_{j=1}^{s}\frac{2}{q_j}=(-1)^{\sum_{j=1}^{s}(q_j{}^2-1)/8}$$

$$=(-1)^{(Q^2-1)/8}.$$

THEOREM 21 : If P and Q are odd and positive and if (P,Q) = 1, then

(17) $$\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right)=(-1)^{((P-1)/2)(Q-1/2)}.$$

PROOF : Writing $P=\prod_{i=1}^{r} p_i$ and $Q=\prod_{j=1}^{r} q_j$

$$\left(\frac{P}{Q}\right)=\sum_{j=1}^{s}\left(\frac{P}{q_j}\right)=\sum_{j=1}^{s}\ \sum_{i=1}^{r}\left(\frac{p_i}{q_j}\right)$$

$$=\left(\frac{P}{Q}\right)\left(\frac{Q}{P}\right)(-1)^{\sum_{j=1}^{s}\sum_{i=1}^{r}\{(p_1-1)/2\}\{(q_j-1)/2\}}$$

By theorem 15 we have

$$\sum_{j=1}^{s}\ \sum_{i=1}^{r}\ \frac{(p_i-1)}{2}\ \ \frac{(q_i-1)}{2}=\sum_{i=1}^{r}\frac{(p_i-1)}{2}\sum_{j=1}^{s}\frac{(q_j-1)}{2}$$

and

$$\sum_{i=1}^{r}\frac{p_i-1}{2}\equiv\frac{P-1}{2},\sum_{j=1}^{s}\frac{q_j-1}{2}\equiv\frac{Q-1}{2}\ (\text{mod } 2)$$

and

$$\left(\frac{P}{Q}\right)=\left(\frac{Q}{P}\right)(-1)^{\{(p-1)/2\}\ (Q-1)/2\}}$$

which proves the theorem.

For example

$$\left(\frac{105}{317}\right)=\left(\frac{317}{105}\right)=\left(\frac{2}{105}\right)=1.$$

Example 1 : Determine whether 888 is a quadratic residue or non-residue of the prime 1999.

Solution : We have

$$\left(\frac{888}{1999}\right)=\left(\frac{4}{1999}\right)\left(\frac{2}{1999}\right)\left(\frac{111}{1999}\right)$$

$$=\left(\frac{111}{1999}\right)=\left(\frac{3}{1999}\right)\left(\frac{37}{1999}\right).$$

We can calculate (3/1999) and (37/1999) by applying quadratic reciprocity law . Also we have by Jacobi symbol

$$\left(\frac{111}{1999}\right)=-\left(\frac{1999}{111}\right)=-\left(\frac{1}{111}\right)=-1.$$

Therefore 888 is a quadratic non-residue modulo 1999.

Example 2 : Evaluate $\left(\frac{595}{7657}\right)$.

Solution : 595 = 5 x 7 x 17, 7657 = 13 x 19 x 31.

Hence (7657, 595) = 1.

$$\left(\frac{595}{7657}\right)=\left(\frac{7657}{595}\right)=\left(\frac{517}{595}\right)\left(\frac{595}{517}\right)=\left(\frac{28}{517}\right)$$

$$=\left(\frac{2}{517}\right)\left(\frac{39}{517}\right)=(-1)\left(\frac{517}{39}\right)=(1-)\left(\frac{10}{39}\right)$$

$$=(-1)\left(\frac{2}{39}\right)\left(\frac{5}{39}\right)=-\left(\frac{39}{5}\right)$$

$$=-\left(\frac{1}{5}\right)=-1.$$

7.6. COMPUTING SQUARE ROOTS MOD p.

Once we know that the congruence

$$x^2 \equiv a \pmod p$$

has a solution, how do we find it ? We describe an algorithm that is extremely fast in its search for a solution.

Case*(i)* If $p \equiv 3 \pmod 4$ ie $p = 4n + 3$, the solution is simple.

For in this case $x = a^{n+1} = a^{(p+1)/4}$ is a solution to $x^2 \equiv a \pmod p$.

We verify this by

$$x^2 \equiv a^{2n+2} \equiv a^{2n+1}\ a \equiv a^{(p-1)/2}\ a \equiv 1\ a \pmod p$$

since $a^{p-1/2} \equiv 1 \pmod p$ for a quadratic residue a.

Case (ii) If $p \equiv 1 \pmod 4$ we can have $p \equiv 1.5 \pmod 8$. If $p \equiv 5 \pmod 8$ and 2 is quadratic nonresidue mod p = 8k+5, we can solve the congruence as follow:

2 is a quadratic residue modulo p when $p = 8k + 5$.

By Euler's criterion

$$a^{p-1/2} \equiv 1 \pmod p$$

$\Rightarrow$ $$(a^{p-1/2} - 1) \equiv 0 \pmod p$$

If $$a^{p-1/4} \equiv 1 \pmod p$$

then $x \equiv a^{p+3/8} \pmod{p}$ is a solution otherwise

$x \equiv 2a\,(4a)^{p-3/8} \pmod{p}$ is a solution.

If $p = 4n + 3$. We write $p-1 = 2s$ with s odd . Let $x = a^{n+1} = a^{(s+1)/2}$ is the solution to $x^2 \equiv a \pmod{p}$ because $(a^{(s+1)/2})^2 \equiv a^s a$ and $a^s \equiv 1 \pmod{p}$. with s odd.

We can not repeat this for $p \equiv 3 \pmod{4}$ as $p-1/2$ is even ie if we write $p-1 = 2s$, then $s+1/2$ is not an integer. We write $p-1 = 2^r s$, with s odd. Let us try $x = a^{(s+1)/2}$ and see how much it differs from a solution

$$x^2 \equiv a^{s+1} \equiv a^s\, a \pmod{p}.$$

Now both a and x^2 are quadratic residues. So a^s is also quadratic residue. Suppose we know an element z such that $z^2 \equiv a^s \pmod{p}$, then we can write

$x^2 \equiv z^2\, a \pmod{p}$ multiplying by the inverse of z^2,

we get $(x\, z^{-1})^2 \equiv a \pmod{p}$

that is, we can solve $x^2 \equiv a$ (mod p) if we can solve $z^2 \equiv a^s \pmod{p}$

To solve $z^2 \equiv a^s \pmod{p}$

The $\text{ord}_p\, a^s | 2^{r-1}$ and if z is a solution then $\text{ord}_p(z) \mid 2^r$.

Then there are 2^r elements of order dividing 2^r, and z must be one of these. Since 2^r can be much smaller than p, to find the solution is simpler. We illustrate this by example.

Example : Solve $x^2 \equiv 2 \pmod{41}$.

We write $p-1 = 41-1 = 40 = 8.5 = 2^3.5$, so $r = 3$ and $s = 5$ (odd). To search for elements of order $2^3 = 8$, the most natural idea is to look for a primitive root g (which has order 40); then g^5 and all its powers will have orders dividing 8. Now 7 is a primitive root modulo 41, and $7^5 \equiv 38 \pmod{41}$ has order 8.

Let $S = \{ 38, 38^2, 38^3, 38^4, 38^5, 38^6, 38^7, 38^8\}$

All the elements of S have order dividing 8, and S includes all elements of orders that divide 8, as there are eight elements of order dividing 8. We reduce S modulo 41 to get

$$S = \{ 38,\ 9,\ 14,\ 40,\ 3,\ 32,\ 27,\ 11\}.$$

Now $2^5 \equiv 32 \equiv 38^6 \pmod{41}$ is in this set

and our solution is

$$x \equiv 2^{(5+1)/2}\, 38^{-6/2}$$

$$\equiv 2^3 \,.\, 38^{-3} \equiv 2^3\, 38^5 \text{ (since } 38^{-3} \equiv 38^5 \pmod{41})$$

$$\equiv 8.3 \equiv 24 \pmod{41}\,.$$

So $x \equiv 24 \pmod{41}$ satisfies the congruence

$$x^2 \equiv 2 \pmod{41}.$$

ANOTHER ALGORITHM FOR FINDING SQUARE ROOTS MODULO p.

Let p be an odd prime, and suppose that we know a quadratic non residue n.

Let a be an integer such that $(a/p) = 1$. We want to find an integer x such that

$$x^2 \equiv a \pmod{p}.$$

First write $p-1 = 2^r s$ where s is odd . Then compute n^s modulo p, and call that b. Next compute $a^{(s+1)/2}$ mod p, and call that α,

Claim : We claim that α is close to being a square root of a Take the ratio α^2/a, we claim that we get 2^{r-1} th root of unity modulo p.

$$(a^{-1}\alpha)^{2^{r-1}} = a^{s\,2^{r-1}} = a^{(p-1)/2} = (a/p) = 1.$$

We must then modify α by a suitable 2^r th root of unity to get an x such that x^2/a is 1. To do this we claim that b is a primitive root of 2^r th root of unity, which means that all the 2^r th root of unity are powers of b . To see this first we note that b is a 2^r th root of 1, because

$b^{2^r} = n^{2^r s} = n^{p-1} = 1$. If b were not primitive, there would be a lower power (a divisor of 2^r) of b that gives 1. But then b would be an even power of a primitive 2^r-th root of unity, so would be a square. This is impossible because $(b/p) = (n/p)^s = -1$ (since s is odd and n is a non residue).

These b is a primitive 2^r-th root of unity. Next to find b^j, $0 < j < 2^r$, such that

$x = b^j\alpha$ gives the desired square root of a. Write j in binary form as $j = j_o + 2j_1, + \ldots + 2^{r-2} j_{r-2}$ and show how one successively determines whether j_o, $j_1, \ldots$ is 0 or 1. To find binary digits of j we have to proceed as follows.

1. Raise (α^2/a) to the 2^{r-2}-th power. The square of this is 1.

Hence we get either ± 1. If we get 1 then $j_o = o$, if we get -1, take $j_o = 1$. We note that j_o has been chosen so that

$((b^{j_o}\alpha)^2/a)$ is a 2^{r-2} th root of unity .

2. Suppose we have found $j_o, j_1, \ldots j_{k-1}$

such that $\dfrac{\left(b^{j_0+2j_1+\dots\dots+2^{k-1}j_{k-1}}\alpha\right)^2}{a}$ is 2^{r-k-1} th root of unity and we want to find j_k. Raise this number to half the power that gives 1, and choose j_k according to whether you get +1 or –1

$$\text{if} \quad \left(\frac{\left(b^{j_0+2j_1,+\dots+2^{k-1}j_{k-1}}\alpha\right)^2}{a}\right)^{2^{r-k-1}} = \begin{cases} 1 \\ -1 \end{cases}$$

then take $j_k = \begin{cases} 0 \\ 1 \end{cases}$, respectively.

$$\text{i.e.} \quad \frac{\left(b^{j_0+2j_1+\dots\dots+2^{k}j_k}\alpha\right)^2}{a} \text{ is a } 2^{r-k-2} \text{ th root of unity.}$$

when we get $k \equiv r\text{-}2$ and find j_{r-2}, we then have

$$\frac{\left(b^{j_0+2j_1,+\dots\dots+2^{r-2}j_{r-2}}\alpha\right)^2}{a} = 1.$$

Example : Find square root of a = 186 mod p; p = 401 i.e. to solve $x^2 \equiv 186 \pmod{401}$ using the above algorithm.

Solution : The first non residue is n = 3. we have p-1 = 400 = 2^4. 25, and so

$$b = 3^{25} = 268 \text{ and}$$

$$\alpha = a^{13} = (186)^{13} \equiv 103 \pmod{401}.$$

Compute $a^{-1} = (168)^{-1} = 235 \pmod{401}$, $\alpha^2/a = 98$, which must be an 8th root of unity. Compute 98^4. Since $98^4 \equiv -1 \pmod{401}$ and so $j_o = 1$.

Next we compute $(b\alpha)^2/a = -1$.

Since the 2nd power of this is 1, we have $j_i = 0$ and hence $j_2 = 1$.

Thus j = 5 and the desired square root is $b^5 \alpha = 304$.

7.7. APPLICATION TO CRYPTOGRAPHY

We have studied public key cryptography. For identification a signature attached to a message can be used to verify the sender's identity. In many applications, it is necessary to verify the identity of the sender before a message is sent . For example we use a Personal Identification Number or PIN to conduct many transactions . Telephone calling cards, and credit cards use a PIN to validate transactions.

Suppose Amit is using a PIN as a secret key to access some services. Some one looking over his shoulder or eavesdropping electronically can receive his PIN and impersonate Amit . Identification schemes are designed to protect against security problems that arise when a secret key is compromised. Instead of revealing the secret key to verify Amit's identify, these scheme provide a mechanism for Amit to prove that he knows the secret key . A proof of identify is based on some computation involving this key and the intermediate results of computation are different for each identification session .

Now one of the identification scheme is based on the difficulty of factoring integers. Let $n = pq$. where p and q are primes solving quadratic congruence modulo n is equivalent to computing the prime factorization of n. Suppose a is a quadratic residue modulo n. If we know the prime factorization of n, then we can solve the congruence $x^2 \equiv a \pmod{n}$ by solving $x^2 \equiv a \pmod{p}$ and $x^2 \equiv a \pmod{q}$. Conversely we can compute the four roots of $x^2 \equiv a \pmod{n}$ then we can factor n. If the roots are $\pm x_0$ and $\pm y_0$ then $x^2_0 \equiv y^2_0 \pmod{n}$ but $x_0 \equiv \pm y_0 \pmod{n}$, so $(x_0 - y_0, n)$ is a proper factor of n.

In this identification scheme due to Feigh–Fiat–Shamir, the number $n = pq$ can be shared among a group of users, without their knowing p and q. A trusted central authority can assign n and public and private keys based on n. We require $(\upsilon, n) \neq 1$. Amit's private key is number s, $0 < s < n$, such that

$$s^2 \equiv \upsilon^{-1} \pmod{n}.$$

The protocol for Amit to prove his identity to Bobby consists of the following steps. Bobby knows the number n and Amit's public key υ.

1. Amit selects a random number r, $o < r < n$ and send $x = r^2 \bmod n$ to Bobby
2. Bobby selects a random bit, b = 0 or 1 and sends b to Amit.
3. If b = 0, then Amit returns r to Bobby, otherwise, Amit returns $y = (rs) \bmod n$.

4. If $b = 0$, then Bobby verifies that $x = r^2 \bmod n$ and if $b = 1$, then Bobby verifies that $y^2 \equiv x\,\upsilon^{-1} \pmod{n}$.

Verification : Amit's identity is verified because he is proving his knowledge of the private key s, without revealing it . This protocol is repeated many time to validate Amit's identity. This scheme is secure, because if a third person Anu is trying to impersonate Amit without knowing s satisfying $s^2 \equiv \upsilon^{-1} \pmod{n}$, then Anu cannot satisfy both the condition in step 4. If he sends x such that $x = r^2 \pmod{n}$, then he can not find $y^2 \equiv x\,\upsilon^{-1} \pmod{n}$ without knowing prime factors of n. If he chooses x and y such that $y^2 \equiv x\,\upsilon^{-1} \pmod{n}$, then he will be able to satisfy the quarry when $b = 1$ but not when $b = 0$. If b is randomly chosen bit, the probability that Anu can guess it correctly in advance is ½. If the protocol is repeated k times, the probability that Anu has guessed each bit correctly is ½ k. So if k is chosen to be large value, say $k \geq 50$, then the chance of some one impersonality Amit are slim.

THEOREM 22 : Suppose x is a quadratic residue modulo $n = pq$, with p and q primes satisfying $p \equiv q \equiv 3 \pmod{4}$. Then there is a unique y such that $y^2 \equiv x \pmod{n}$ and y is a quadratic residue modulo n.

PROOF : Since n is a product of two primes, there are four solutions to the congruence $y^2 \equiv x \pmod{n}$. If y_o is a solution, then the four solutions are

$$\begin{array}{ll}
y \equiv y_o \pmod{p} & y \equiv y_o \pmod{p} \\
y \equiv y_o \pmod{q} & y \equiv -y_o \pmod{q} \\
y \equiv -y_o \pmod{p} & y \equiv -y_o \pmod{p} \\
y \equiv y_o \pmod{q} & y \equiv -y_o \pmod{q}.
\end{array}$$

The four possible solutions give rise to four possible combinations of the Legendre symbol (y/p) and (y/q). The four pairs are:

$\{(y_o/p), (y_o/q) \quad , \quad (y_o/p), (-y_o/q)\}$,

$(-y_o/p), (y_o/q), (-y_o/p), (-y_o/q)$

Since $p \equiv q \equiv 3 \pmod{4}$, the Legendre symbols $(-1/p)$ and $(-1/q)$ are both -1 , and each of the four possible combinations of signs. $\{1, 1\}$, $\{1, -1\}$, $\{-1, 1\}$ and $\{-1, -1\}$ must occur once, irrespective of the values of (y_o/p) and (y_o/q), Hence, there is only one y such that $(y/p) = 1$ and $(y/q) = 1$, that is there is only one solutions to $y^2 \equiv x \pmod{n}$ such that y is a quadratic residue modulo n.

Example : Consider $n = pq$ with $p = 331$ and $q = 431$. We select $x = 115.35$ as a quadratic residue modulo n we can solve $y^2 \equiv x \pmod{n}$, to obtain the solutions $y \equiv \pm 127060, \pm 58962 \pmod{n}$. Of these four, only one, 127060 is a quadratic residue modulo n.

As an application to cryptography to construct a one-time pads in a public key system. A onetime pad is a string $s_0, s_1, s_2 \dots s_k$ that is added to the plaintext $p_0, \dots p_k$ to produce a ciphertext $c_0, c_1, .. c_k$. This is represented by $c_i = p_i + s_i$ (mod 2). Suppose Amit set a public key n = pq, where p and q satisfy $p \equiv q \equiv 3$ (mod 4). If Bobby wishes to send a coded message to Amit, then he chooses a number x_o and compute the sequence $x_o, x_1 \dots x_k \, x_{k+1}$. where $x_{i+1} \equiv x_i^2$ (mod n) for i = 0, 1, 2, ..k then Bobby compute the bits $s_i = x_i$ (mod 2). The random bit stream $s_1, s_2 .. s_k$ is used as one time pad to the plaintext $p_1, p_2 .. p_k$.

Bobby sends the ciphertext $c_0, c_1 .. c_k$ with $c_i = p_i + s_i$ (mod 2) and the integer x_{k+1} Amit uses x_{k-1} to recover the number $x_k, x_{k-1}, \dots x_1$ and from this, the one time pad . It is easy to recover the plaintext from the knowledge of the ciphertext and the one time pad.

The scheme works because there is only one square root of each term that is a quadratic residue. The scheme is more secure as it depends on computing y satisfying $y^2 \equiv x$ (mod n) is equivalent to factoring n

EXERCISES

1. Prove that $\sum_{j=1}^{p-1} (j/p) = 0$, p an odd prime.

2. Evaluate (*i*) $\left(\frac{-25}{83}\right)$ (*ii*) $\left(\frac{-23}{83}\right)$ (*iii*) $\left(\frac{51}{79}\right)$

   (*iv*) $\left(\frac{71}{73}\right)$ (*v*) $\left(\frac{105}{73}\right)$ (*vi*) $\left(\frac{-35}{97}\right)$.

3. Prove that if p and q are distinct prime of the form 4k + 3, and if $x^2 \equiv p$ (mod q) has no solution, then $x^2 \equiv q$ (mod p) has two solutions.
4. Show that 665 is a quadratic residue of 1443.
5. Find the solutions of $x^{78} \equiv 1$ (mod 13)
6. Let p be an odd prime prove each of the following statements:

   (*i*) $\sum_{r=1}^{p-1} r \left(\frac{r}{p}\right) = 0$ if $p \equiv 1$ (mod 4).

   (*ii*) $\sum_{r=1}^{p-1} r = \frac{p(p-1)}{4}$ if $p \equiv 1$ (mod 4).

   $\left(\frac{r}{p}\right) = 1$

(iii) $\sum_{r=1}^{p-1} r^2 \ (r/p) = p \sum_{r=1}^{p-1} r\ (r/p)$ if $p \equiv 3 \pmod 4$.

(iv) $\sum_{r=1}^{p-1} r^3 \ (r/p) = 3/2\ p \sum_{r=1}^{p-1} r^2\ (r/p)$ if $p \equiv 1 \pmod 4$.

(v) $\sum_{r=1}^{p-1} r^4\ (r/p) = 2p \qquad \sum_{r=1}^{p-1} r^3\ (r/p) - p^2 \sum_{r=1}^{p-1} r^2\ (r/p)$

if $p \equiv 3 \pmod{4}$.

7. Prove that if $p \equiv 3 \pmod 4$

$$\{(2/p) - 2\} \sum_{r=1}^{p-1} r\ (r/p) = p \sum_{r=1}^{q} (r/p)$$

where $q = p\text{-}1/2$.

8. Prove that 3 is a quadratic non residue of all primes of the form $2^{2n} + 1$, as well as primes of the form 2^p- 1, where p is a prime.

9. Determine whether the following quadratic congruences are solvable.

(a) $x^2 \equiv 219 \pmod{419}$.

(b) $3x^2 + 6x + 5 \equiv 0 \pmod{89}$.

(c) $2x^2 + 5x - 9 \equiv 0 \pmod{101}$.

10. Evaluate the Jacobi symbols

(a) $\left(\frac{21}{221}\right)$ *(b)* $\left(\frac{215}{253}\right)$ *(c)* $\left(\frac{631}{1099}\right)$

11. Use Gauss lemma to show that 17 is a quadratic residue modulo 19.

12. Does the congruence $x^2 \equiv 631 \pmod{1093}$ has any solution ?

13. Show that $x^2 \equiv 15 \pmod{89}$ has no solution.

14. Write a computer program to evaluate Legendre symbol (a/p) using Euler's criterion.

15. Show that the smallest positive quadratic non-residue modulo p is always a prime.

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CHAPTER

8

PRIMALITY TESTING AND FACTORING

8.1. PSEUDOPRIMES AND CARMICHAEL NUMBERS

As an application of number theory to cryptography we want to know if large number n is prime. For example , we have shown in chapter v that in the RSA public key cryptosystem we need to find a large "random" prime. For this we use primality testing which means to determine whether an integer of a certain very special type is a prime.

PRIMALITY TEST : A primality test is a criterion for a number n not to be prime. If n 'passes' a primality test, then it may be prime. If it passes a whole lot of primality tests, then it is very likely to be prime. If n fails any single primality test then n is surely composite. But if n is composite and large enough then it is difficult to factorize it. In this chapter we will study some primality test and factorization of big primes.

The simplest primality test is "trial division". Suppose n is a large odd integer. To test whether or not n is prime. Take an odd integer m and see whether or not it divides n. If $m \neq 1$, and $m \mid n$, then n is composite otherwise n passes the primality test "trial division by m". If n passes all the trial division test then n is prime. We know that n is prime when m reaches $\sqrt{n}$. Of course this trial division method is very time consuming. We will search for other test. One is the Fermat's little theorem Fermat's theorem tells that if gcd (b,n) = 1 for any b and n is prime then

(1) $b^{n-1} \equiv 1 \pmod{n}$.

If n is not prime it is still probable that (1) holds. We define pseudoprime as follow:

DEFINITION : If n is an odd composite number and b is an integer such that gcd $(b,n) = 1$ and $b^{n-1} \equiv 1 \pmod{n}$ then n is called as the pseudoprime to the base b.

We say that 'pseudoprime' is a number n that 'pretends' to be prime by passing the test (1).

Example 1 : $n = 91$ is a pseudoprime to the base $b \equiv 3$, because

gcd. $(3, 91) = 1$ and

$$3^{90} \equiv 1 \pmod{91}.$$

But 91 is not a pseudoprime to the base 2, because $2^{90} \equiv 64 \pmod{91}$.

But 91 is composite and

$$2^{90} \equiv 1 \pmod{91}.$$

Therefore 91 is not a pseudoprime to the base 2.

Example 2 : 15 is a pseudoprime to the base 4 and 11 but not to the base 2 and 3.

We have to show that

$$4^{14} \equiv 1 \pmod{15}$$

Since $$4^2 \equiv 1 \pmod{15}$$

$\Rightarrow$ $$4^{14} \equiv 1 \pmod{15}$$

$$11^{14} \equiv 1 \pmod{15} \text{ since } 11^2 \equiv 1 \pmod{5},$$

$$11^2 \equiv 1 \pmod{3}$$

Hence $$11^2 \equiv 1 \pmod{15}$$

$\Rightarrow$ $$(11)^{14} \equiv 1 \pmod{15}.$$

But $$2^{14} \equiv 1 \pmod{15} \text{ and } 3^{14} \equiv 1 \pmod{15}.$$

THEOREM 1 : Let n be an odd composite integer. Then

(*a*) n is pseudoprime to the base b, where gcd $(b.n) = 1$, if and only if the order of b divides $n-1$.

(*b*) If n is pseudoprime to the bases b_1 and b_2 where gcd $(b_1,n) = 1$ and gcd $(b_2,n) = 1$, then n is a pseudoprime to the base $b_1 b_2$ and also to the base $b_1b_2^{-1}$.

(*c*) If n fails the test (1) for a single base b, then n fails (1) for atleast half of the possible bases.

PROOF : To prove (*a*): From the hypothesis we have $b_1^{n-1} \equiv 1 \pmod{n}$ and

$$b_2^{n-1} \equiv 1 \pmod{n}.$$

These two imply $(b_1\, b_2)^{n-1} \equiv 1 \pmod{n}$. This proves (*a*).

Similarly the proof of (*b*) follows . To prove (*c*), let $\{b_1, b_2, .. b_s\}$ be the set of all bases for which n is pseudoprime, i.e. the set of all integers $0 < b_i < n$ for which the congruence (1) holds. Let b be a fixed base for which n is not a pseudoprime. If n were pseudoprime for any of the bases $b\, b_i$, then by (*b*) it would be a pseudoprime for the base $b \equiv (b\, b_i)\, b_i^{-1} \pmod{n}$ which is not the case. Thus, for the s distinct residues $\{ bb_1, bb_2, \ldots bb_s\}$, the integer n fails the test (1). Hence, there are at least as many bases for which n fails to be a pseudoprime as there are base for which (1) holds . Hence proved.

Now the question is: For a composite n (1) holds for every b ?

The answer is yes, and such a number is called a Carmichael number. Such numbers first studied by R. D Carmichael in 1912. We define Carmichael number as follows :

DEFINITION 2 : A Carmichael number is a composite integer n such that

(2) $$b^{n-1} \equiv 1 \pmod{n} \text{ for every b.}$$

Example : 561 is the smallest Carmichael number.

561 is composite as 561 = 3. 11. 17. To show that $a^{560} \equiv 1 \pmod{561}$.

By Fermat's theorem

$$a^2 \equiv 1 \pmod{3}$$

$$\Rightarrow \quad a^{560} \equiv (a^2)^{280} \equiv 1 \pmod{3}$$

$$a^{10} \equiv 1 \pmod{11}$$

$$\Rightarrow \quad a^{560} \equiv 1\ (a^{10})^{56} \equiv 1 \pmod{11}.$$

$$a^{16} \equiv 1 \pmod{17}$$

$$\Rightarrow \quad a^{560} \equiv (a^{16})^{35} \equiv 1 \pmod{17}.$$

Since 3 , 11, 17 are mutually relatively prime we obtain

$$a^{560} \equiv 1 \pmod{561} \text{ for all a.}$$

Example 2 : 341 is not a Carmichael number.

$341 = 11 \times 31$

$a^{10} \equiv 1 \pmod{11}$ $\qquad$ $a^{30} \equiv 1 \pmod{31}$

$(a^{10})^{34} \equiv 1 \pmod{11}$ $\qquad$ $a^{340} \equiv 1 \pmod{31}$

$a^{340} \equiv 1 \pmod{11}$ $\qquad$ So $\qquad$ $a^{340} \equiv 1 \pmod{341}$

PROPERTIES OF CARMICHAEL NUMBERS

THEOREM 2 : A composite number n is a Carmichael number if and only if for every $p|n$ we have that $p-1|\ n-1$.

PROOF : First suppose that $p-1|\ n-1$ for every p dividing n. Let 3 be any base, where gcd $(b,n) = 1$. Then for every p dividing n we have b^{n-1} is a power of b^{p-1}, and so is $\equiv 1 \pmod p$. These $b^{n-1} - 1$ is divisible by all prime factors p of n, and hence by their product which is n . Hence (2) holds. Conversely, suppose that there is a p such that $p-1 \ | \ n-1$. Let g be an integer which is a generator. Find an integer b which satisfies $b \equiv g \pmod p$ and $b \equiv 1 \pmod{n/p}$. Then gcd $(b, n) = 1$, and $b^{n-1} \equiv g^{n-1} \pmod p$. But $g^{n-1} \equiv 1 \pmod p$ because $(n - 1)$ is not divisible by the order $p-1$ of g. Hence $b^{n-1} \equiv 1 \pmod p$ and so (2) can not hold This completes the proof of the theorem.

THEOREM 3 : Let n be a composite integer . If n is divisible by a perfect square > 1 then n is not a Carmichael number.

PROOF : Suppose that $p^2|n$. Let g be a generator modulo p^2, i.e. an integer such that $g^{p(p-1)}$ is the lowest power of g which is congruent to 1 mod p^2. Let n' be the product of all primes other than p which divide n. By the Chinese Remainder theorem, there is an integer b satisfying the congruence.

$$b \equiv g \pmod{p^2} \text{ and } b \equiv 1 \bmod n'$$

Then b is, like g, a generator modulo p^2 , and it satisfies gcd $(b, n) = 1$. since it is not divisible by p or by any prime which divides n' n is not a pseudoprime to the base b. If (2) holds, then since p^2 n, we have $b^{n-1} \equiv 1 \pmod{p^2}$. But in that case $p(p-1)|n-1$, since $p(p-1)$ is the order of b mod p^2. however $n-1 \equiv -1 \bmod p$ since $p|n$ and this gives that $n-1$ is not divisible by $p(p-1)$. This contradiction proves that there is a base b for which n fails to be a pseudoprime.

THEOREM 4 : A Carmichael number is the product of at least three distinct primes.

PROOF : Suppose $n = p\,q$ the product of two distinct primes. Suppose that $p < q$.

Then , if n were a Carmichael number, we would have $n-1 \equiv 0 \pmod{q-1}$.

But $n-1 = p\,(q-1+1) - 1 \equiv p-1 \pmod{2-1}$ and this is not $\equiv 0 \pmod{q-1}$,

since $0 < p-1 < q-1$. This completes the proof.

Example 1 : 561 is a Carmichael numbers since $n = 561 = 3.\ 11.\ 17$

and 560 is divisible by 3–1, 11–1 and 17–1.

Example 2 : 1105 = 5.13.17 is a Carmichael number since 1104 is divisible by 4, 12 and 16.

8.2. STRONG PSEUDOPRIMES AND PROBABILISTIC PRIMALITY TESTING.

We define strong pseudoprimes as follow :

DEFINITION 1 : Let n be an odd composite number, and write $n-1 = 2^s t$ with t odd. Let b be any integer. If n and b satisfy the condition either $b^t \equiv 1$ (mod n) or there exist r, $0 \leq r < s$, such that $b^{2^r t} \equiv -1$ (mod n), then n is called a strong pseudoprime to the base b.

DEFINITION 2 : Let n be an odd integer, and let (b/n) denote the Jacobi symbol. If n is a prime number, then

$$(3) \qquad b^{(n-1)/2} \equiv (b/n) \bmod n.$$

DEFINITION 3 : If n is an odd composite number and b is an integer such that gcd(n, b) = 1 and (3) holds, then n is called a Euler pseudoprime to the base b.

THEOREM 5 : if n is a Euler pseudoprime to the base b, then it is pseudoprime to the base b. But the converse is not true.

PROOF : If n is a Euler pseudoprime to the base b, by definition

$$b^{(n-1)/2} \equiv (b/n) \pmod n, \text{ g. c. d } (n, b) = 1.$$

Squaring both sides of the congruence we have

$$b^{n-1} \equiv 1 \pmod n; \text{ gcd } (n, b) = 1.$$

Which implies n is a pseudoprime to the base b.

The converse is not true can be shown by the following example .

Example : 91 is a pseudoprime to the base 3. However $3^{91-1/2} \equiv 27$ (mod 91) so (3) is not true for n = 91 and b = 3. Hence 91 is not Euler pseudoprime to the 3. But 91 is a pseudoprime to the base 10, since $10^{91-1/2} = 10^{45} \equiv 10^3 \equiv -1$ (mod 91) and since

$$\left(\frac{10}{91}\right) = \left(\frac{2}{91}\right)\left(\frac{5}{91}\right).$$

$$\left(\frac{2}{91}\right) - (-1)^{(91)^2-1/8} = -1.$$

$$\left(\frac{5}{91}\right)=\left(\frac{91}{5}\right)(-1)^{(5-1)91-1)/4}=\left(\frac{91}{5}\right)=\left(\frac{1}{5}\right)-$$

Hence $\left(\frac{10}{91}\right)=-1.$

So $10^{45}\equiv\left(\frac{10}{91}\right)$ (mod 91).

THEOREM 6 : If $n \equiv 3 \bmod 4$, then n is strong pseudoprime to the base b if and only if it is an Euler pseudoprime to the base b .

PROOF : Since $n \equiv 3 \pmod 4$ i.e. $n = 4k + 3$,

here $n-1 = 4k + 2 = 2(2k+1)$

Hence $s = 1$ and $t \equiv n-1/2$.

We see that n is a strong pseudoprime to the base b if and only if $b^{(n-1)/2} \equiv \pm 1 \pmod n$. If n is a Euler pseudoprime then the congruence holds by definition. Conversely suppose that $b^{n-1/2} \equiv \pm 1$. To show that ± 1 on the right is the Jacobi symbol (b/n).

But for $n \equiv 3 \pmod 4$ i.e. $n = 4k + 3$, $\pm 1 = (\pm 1/n)$, and so

$$\left(\frac{b}{n}\right)=\left(\frac{b(b^2)^{n-3/4}}{n}\right)=\left(\frac{b^{n-1/2}}{n}\right)$$

$$\equiv b^{n-1/2} \pmod n.$$

Hence $b^{n-1/2} \equiv (b/n) \pmod n$.

Hence n is Euler pseudoprime. This completes the proof of the theorem.

THEOREM 7 : If n is a strong pseudoprime to the base b, then it is an Euler pseudoprime to the base b.

The proof is difficult and lengthy. We are leaving it.

THEOREM 8 : If n is an odd pseudoprime to the base 2, then the Mesenne number 2^n-1 is a strong pseudoprime to the base 2.

PROOF : First we will prove that $2^n - 1$ a pseudoprime to the base 2.

Let $n^1 = 2^n - 1$.

To show that

$$2^{n1}-1 \equiv 1 \pmod{n^1} \text{ or } 2^{n1} \equiv 2 \pmod{n^1}.$$

n is a 2 pseudoprime implies

$$2^{n-1} \equiv 1 \pmod n \text{ or } 2^n \equiv 2 \pmod n$$

or $n | 2^n - 2 = n^1 - 1.$

$\Rightarrow$ $n^1 - 1 = nk$ for some k, then $2^{n^1-1} - 1 = 2^{nk} - 1$

$$= (2^n - 1)(2^{n(k-1)} + ... + 2^n + 1),$$

$$= n^1 (2^{n(k-1)} + ... + 2^n + 1)$$

$\Rightarrow$ $n^1 | 2^{n^1-1} - 1$ or $2^{n1-1} \equiv 1 \pmod{n^1}$

or $2^{n1} \equiv 2 \pmod{n^1}$ the required result.

Next to show that $2^n - 1$ is a strong pseudoprime to the base 2.

We write $n^1 - 1 = 2^n - 2 = 2(2^{n-1} - 1) = 2\,nk$, so the odd part of $n^1 - 1$ is nk.

Clearly $2^n - 1 \equiv 0 \pmod{2^n - 1}$ or $2^n \equiv 1 \pmod{n^1}$.

This implies that $2^{nk} \equiv 1 \pmod{n^1}$ since nk is odd part of $n^1 - 1$, this shows that

$n^1 = 2^n - 1$ is a strong pseudoprime to the base 2 This completes the proof.

ALGORITHM 1 : (Simple primality Test)

Given $n \leq 25.10^9$, this agorithm determines if n is prime.

1. If n fails the strong pseudoprime test to base 2, then n is composite.
2. If n fails strong pseudoprime test to base 3, then n is composite
3. If n fails the strong pseudoprime test to base 5, then n is composite.
4. If n fails the strong pseudoprime test to the base 7 then n is composite.
5. If n fails strong pseudoprime test to the base 11 then n is composite.

Example : Consider n = 117371. We factorize n–1 = 2.58685.

Compute

$2^t \equiv -1 \pmod{n}$ where t = 58685,

and $2^{2t} \equiv 1 \pmod{n}$

and $3^t \equiv 1 \pmod{n}$ where t = 58685.

This implies n is strong pseudoprime

Note : It is to note that the strong pseudoprimes are much more useful then pseudoprimes to detect compositeness.

If n fails a strong pseudoprime test, then it is composite. Suppose we check for 50 random bases, then the probability that a composite number successfully passes 50 random pseudoprime test is less than ½ 50. In this case the number is very likely to be prime. We call a number that passes this test a **probable prime** we must note that this does not prove that the number is prime.

We are giving now the following probabilistic algorithm for compositeness, or detecting probable primes.

ALGORITHM 2 : (Rabin –Miller Probabilistic Primality Test)

We use the strong pseudoprime test to several random bases to check if n is a probable prime. The test should only be applied after checking if the number has any small factors.

1. [initialize] Let $t = n-1$, $s = 0$, $k = 50$
2. [Compute $n-1 = 2^s.t$]. If $t \equiv 0 \pmod 2$, $t = t/2$, $s = s + 1$, and repeat step 2. Otherwise go to step 3.
3. Choose a random integer $b < n$ and set $b = b^t \bmod n$ and $c = 0$. if $b = 1$. go to step 5, otherwise go to step 4.
4. If $c > s-2$ or $b = 1$, then n is composite. If $b = -1$ go to step 5 otherwise

   $b = b^2 \bmod n$, $c = c + 1$, repeat step 4.
5. Let $k = k-1$. If $k > 0$, go to step 3. Otherwise, report that n is a probable prime.

Probabilistic primality tests are quite sufficient for large primes. We give now a short and easy primality test to verify whether n is prime or composite.

We have defined earlier the order and primitive root of an integer modulon.

8.3. PRIMALITY TESTING (APPLICATION OF PRIMITIVE ROOTS AND INDICES TO CRYPTOGRAPAY)

Now we will give Lucas–Lehmer primality test and the ElGamal Cryptosystem.

THEOREM 9 : (Lucas Lehmer) Suppose there exists an integer b such that

$b^{n-1} \equiv 1 \pmod n$. But for each prime q dividing n–1, $b^{(n\ 1)/q} \equiv 1 \pmod n$, then n is prime.

PROOF : To prove that order $_n(b) = n-1$. The congruence $b^{n-1} \equiv 1$ (mod n)

$\Rightarrow \qquad \text{order}_n(b) \mid n-1.$

Let $n-1 = \text{ord}_n\, b.k$ for some k. To show that $k = 1$, so suppose that $k > 1$ and a prime q divide k, then $q \mid n-1$, and we can write

$$b^{(n-1)/q} = b^{(\text{ord}_n b.k)/q} \equiv 1 \pmod n.$$

This contradicts the hypothesis of the theorem, so $k = 1$ and $\text{ord}_n(b) = n-1$.

As $\text{ord}_n (b) \mid \varphi(n)$, we must have $\varphi (n) \geq n-1$, but $\varphi (n) \leq n-1$, therefore $\varphi(n) = n-1$ and n is prime.

Example 1 : Consider $n = 29$. $n-1 = 28 = 2^2.7$

Take $b = 2$. Now

$$2^{n-1} = 2^{28} \equiv 1 \pmod{29}$$

$$2^{28/2} = 2^{14} \equiv 28 \pmod{29}$$

i.e. $$2^{14} \not\equiv 1 \pmod{29}$$

$$2^{28/7} = 2^4 \equiv 16 \pmod{29}. \not\equiv 1 \pmod{29}$$

So by theorem 9, 29 is prime.

Example 2 : Let $n = 911$.

$$n - 1 = 910 = 2.5.7.13$$

Now

$$7^{n-1} \equiv 1 \pmod{n} \qquad 7^{(n-1)/2} \equiv -1 \pmod{n}$$

$$3^{n-1} \equiv 1 \pmod{n} \qquad 3^{(n-1)/5} \equiv 482 \pmod{n}$$

$$2^{n-1} \equiv 1 \pmod{n} \qquad 2^{(n-1)/7} \equiv 568 \pmod{n}$$

$$2^{n-1} \equiv 1 \pmod{n} \qquad 2^{(n-1)/13} \equiv 1 \pmod{n}.$$

The ElGamal system (Application of Discrete logarithm to Cryptography).

The ElGamal system, a publickey cryptosystem, is based on the presumed difficulty of discrete logarithm problem. We know that if $r^x \equiv y \pmod{p}$ then x is the discrete logarithm of y mod p. There is no method to compute x with the knowledge of r, y and p, where p is a large prime.

The ElGamal system can be used for both encryption and signatures. We will describe the method as follows:

Suppose Ajit wishes to receive encrypted message. He chooses a prime p and a primitive root r modulo p. He selects a private key a, $0 < a < p-1$, and computes $b = r^a \bmod p$, Ajit's public key is $k = (r, b, p)$. To encrypt a message m, $0 < m < p$, one has to proceed as follows :

1. Choose a random number s, $1 < s < p-1$
2. compute $y_1 = r^s \bmod p$ and $y_2 = m\, b^s \bmod p$.
3. The ciphertext is $E_K (m) = (y_1,y_2)$.

The security lies in the choice of random number r. Ajit can recover m using the decryption function

$$D_k (y_1, y_2) = y_2 (y_1)^{-a} \bmod p.$$

This is valid because

$$\begin{aligned} D_k(y_1, y_2) &= y_2 (y_1)^{-a} \bmod p \\ &= y_2 (r^{-sa}) \bmod p \\ &= y_2 (r^a)^{-s} \bmod p \\ &= m\, b^s\, b^{-s} \pmod p \\ &= m \pmod p. \end{aligned}$$

Example : Let $p = 37$, $r = 2$ and $a = 31$.

Then $b = 2^{31} \pmod{37} = 22$. Suppose plaintext is $m = 19$. Let A chooses a random number say $s = 7$; then

$$\begin{aligned} y_1 &= 2^7 \pmod{37} \\ &\equiv 17, \\ y_2 &\equiv 19 \,.\, 22^7 \pmod{37} \\ &\equiv 19.2 \pmod{37} \\ &\equiv 1. \end{aligned}$$

Then A sends $E_K(m) = (17, 1)$ to B. To decipher this B computes D_K (17, 1)

$$= 17^{-31} \pmod{37} = 19.$$

The security of the system lies in the computation of the private key a from the public key parameters using discrete logarithm problem, which is difficult. Hence the security of the ElGamal system is equivalent to the discrete log problem.

Another quality of the system which is important for its security is the choice of random number s. The same value of s should not be used with different plaintext. If the same s is used for different plaintext, then it will be possible to recover all plaintext from the knowledge of one.

In the subsequent section we will discuss some factorization method.

8.4. POLLARD'S (p–1) METHOD

We discuss two factorization method due to J. N. Pollard. Pollard's (p–1) methods is the following :

Suppose n is the number to be factored, and say p|n, p is a prime.

Now $b^{p-1} \equiv 1 \pmod p$ for any b such that g cd d (b, p) = 1.

Suppose p – 1 divides a number M; then $a^M \equiv 1 \pmod p$ ie $p | a^M - 1$.

Since p|n and $p | a^M - 1$, p will divide their gcd (n, $a^M - 1$).

We compute $a^M-1 \pmod n$ and $(a^M-1 \bmod n, n)$. If this gcd is not equal to n, then we would have found a non trivial factor of n. This factor is p. Let us illustrate by example.

Example 1 : Consider n = 1073 = 29.37. If p = 29, p–1 = 28. Let b =2

$$2^{28} \equiv 900 \pmod{1073} \text{ and } (900-1, 1073) = 29.$$

Similarly

$$2^{36} \equiv 777 \pmod{1073}$$

and $(777-1, 1073) = 37$, the second factor.

8.5. POLLARD'S RHO METHOD

The first step in this method is to choose an easily evaluated map from Z/nZ to itself, a simple polynomial with integer coefficients such as $f(x) = x^2 +1$. Next choose a particular value $x = x_o$ (x_o = 1 or 2 or randomly generated integer) and compute the successive iterates of f : $x_1 = f(x)$, $x_2 = f(f(x_o))$, $x_3 = f(f(f(x_o)))$.

We define $x_{j+1} = f(x_j)$ j = 0, 1, 2...

Compare between different x_j's, find two which are in different residue classes modulo n but in the same residue classes modulo some divisor of n.

Finding such x_j, x_k, we have gcd $(x_j - x_k, n)$ is equal to a proper divisor of n, and complete the factorization.

Example 1 : Factor 91 choosing $f(x) = x^2 + 1$, $x_o = 1$.

We have $x_1 = f(x_0) = 2$

$$x_2 = 5, x_3 = 26$$

We find gcd $(x_3 - x_2, n)$ = g c d (21, 91) = 7, so 7 is a factor.

Exercise : Factor 4087 using $f(x) = x^2 + x + 1$ and $x_o = 2$.

8.6. FERMAT'S FACTORIZATION METHOD

Fermat factorization is based on the fact that n is equal to difference of two squares, one of which is very small.

THEOREM 10 : Let n be a positive odd integer. There is 1 to 1 correspondence between factorization of n in the form n = ab, where $a \geq b > 0$, and representation of n in the form $t^2 - s^2$, where s and t are non negative integers. The correspondence is given by the equation .

$$t = a+b/2,\ s = a-b/2,\ a = t + s\ b = t - s.$$

PROOF : We can write

$$n = ab = ((a + b)/2)^2 - ((a - b)/2)^2,$$

So we obtain the representation as a different of two square.

Conversely, given $n = t^2 - s^2$, we can write $n = t^2 - s^2 = (t + s)\,(t - s)$.

The equation gives the 1 to 1 correspondence between the two ways of writing n.

If $n = ab$ with a and b close together, then $s = a - b/2$ is small and so t is slightly larger than $\sqrt{n}$. So we find a and b by trying all values for t starting with $[\sqrt{n}] + 1$ until we find $t^2 - n = s^2$ is a perfect square.

Example : Factor 200819

Solution : We have $\sqrt{200819} + 1 = 449$. Now $449^2 - 200819 = 782$ which is not a perfect square. Next we try

$$t = 450 \,.\, 450^2 - 200819 = 1681 = 41^2.$$

Then $200819 = 450^2 - 41^2 = (450 + 41)\,(450 - 41) = 491.\,409$.

For finding a and b we try to find

$$t = [\sqrt{n}] + 1, [\sqrt{n}] + 2 \ldots$$

Also we choose a small k, successively set $t = [\sqrt{kn}] + 1, [\sqrt{kn}] + 2, \ldots$ until we obtain a t for which $t^2 - kn = s^2$ is a perfect square.

Exercise : Factor 141467.

SOME COMPUTER PROGRAM FOR FINDING PRIME FACTORS OF ANY INTEGER.

The following BASIC program will find the prime factors of any integer. The integer is input to the program as data. the program terminates execution whenever a zero is typed as input.

```
100     REM PRIME FACTORS OF ANY INTEGER
110     PRINT "PRIME FACTORS OF ANY INTEGER"
120     PRINT
130     PRINT
140     PRINT
150     PRINT "NUMBER TO BE FACTORED IS";
160     INPUT A
170     IF ABS (A) < = 1 THEN 340
180     LET N = INT (ABS(A))
190     REM FIND AND PRINT PRIMES
200     LET B = 0
210     FOR I = 2 TO N/2
```

```
220         IF N/1 > INT (N/1) THEN 300
230         LET B = B + 1
240         IF B > 1 THEN  260
250         PRINT "PRIME FACTORS OF "; N; "ARE"
260         PRINT 1 ;
270         LET N = N/I
280         IF N =1 THEN 120
290         LET I = I –1
300         NEXT I
310         IF N <> INT (A) THEN 120
320         PRINT N; " IS A PRIME NUMBER"
330         GOTO 130
340         END
```

RUN

```
PRIME FACTORS OF ANY INTEGER
NUMBER TO BE FACTORED IS ? 56
PRIME FACTORS 56 ARE
2          2       2       7
NUMBER TO BE FACTORED IS ? 346
PRIME FACTORS OF 346 ARE
2          173

           NUMBER TO BE FACORED IS ? 397
                   397 IS A PRIME NUMBER

           NUMBER TO BE FACTORED IS ? 560
           PRIME FACTORS OF 560 ARE
           2       2       2       2       5       7
```

The following BASIC program uses Fermat's method to compute the largest factor of a given integer.

```
100         REM LARGEST FACTOR OF ANY NUMBER
110         PRINT "WHAT IS THE NUMBER";
120         INPUT N
130         IF N = 0 THEN 280
140         LET W = INT (SQR (N))
```

```
150         LET X = 2 * W+1
160         LET Y = 1
170         LET  R= W* W–N
180         IF R = 0 THEN 250
190         IF R> 0 THEN 220
200         LET R = R +X
210         LET X = X+2
220         LET R = R–Y
230         LET Y = Y+ 2
240         GOTO 180
250         LET F = (X–Y)/2
260         PRINT “LARGEST FACTOR OF “; N;”IS”; F
270         GOTO 110
280         END
```

RUN

```
WHAT IS THE NUMBER? 311
LARGEST FACTOR OF 311 IS 1
WHAT IS THE NUMBER ? 45
LARGEST FACTOR OF 45 IS 5
WAT IS THE NUMBER ? 0
```

The BASIC program shown below computes the largest factor of the number listed in DATA statement.

```
100         REM LARGEST FACTOR PROGRAM
110         READ N
120         FOR D = 2 TO SQR (N)
130         IF N/D = INT (N/D) THEN 170
140         NEXT D
150         PRINT N, “IS A PRIME NUMBER”
160         GOTO 110
170         PRINT N/D, “IS THE LARGEST FACTOR OF “;N
180         GOTO 110
```

```
190      DATA 3394, 5799, 2827, 1907, 9115
200      DATA 2807, 1495, 373, 19, 206
210      END
```

RUN

```
1697     IS THE LARGEST FACTOR OF      3394
1933     IS THE LARGEST FACTOR OF      5799
257      IS THE LARGEST FACTOR OF      2827
1907     IS A PRIME NUMBER
1823     IS THE LARGEST FACTOR OF      9115
401      IS THE LARGEST FACTOR OF      2807
299      IS THE LARGEST FACTOR OF      1495
373      IS A PRIME NUMBER
19       IS A PRIME NUMBER
103      IS THE LARGEST FACTOR OF 206
```

OUT OF DATA IN LINE 110

To find all pairs of factors of an integer, use the following BASIC program.

```
100      REM PAIRS OF FACTORS OF AN INTEGER
110      PRINT " PAIRS OF FACTORS"
120      PRINT
130      PRINT
140      PRINT "TYPE THE INTEGER";
150      INPUT X
160      PRINT
170      PRINT "THE PAIRS OF FACTORS OF ";X;"ARE:"
180      FOR A = 1 TO SQR (ABS (X))
190      IF INT (X/A) <> X/A THEN 210
200      PRINT A, X/A
210      NEXT A
220      PRINT
230      PRINT
```

```
240        PRINT "TYPE 1 TO STOP; 2 TO CONTINUE";
250        INPUT T
260        IF T <> 1 THEN 120
270        END
```

RUN

```
PAIRS OF FACTORS
TYPE THE INTEGER? 8960
THE PAIRS OF FACTORS OF 8960  ARE :

            1                         8960
            2                         4480
            4                         2240
            5                         1792
            7                         1280
            8                         1120
           10                          896
           14                          640
           16                          560
           20                          448
           28                          320
           32                          280
           35                          256
           40                          224
           56                          160
           64                          140
           70                          128
           80                          112
TYPE 1 TO STOP ; 2 TO CONTINUE?2
TYPE THE INTEGER ? 4680
```

```
THE PAIRS OF FACTORS OF  4680 ARE :
                1                    4680
                2                    2340
                3                    1560
                4                    1170
                5                     936
                6                     780
                8                     585
                9                     520
               10                     468
               12                     390
               13                     360
               15                     312
               18                     260
               20                     234
               24                     195
               26                     180
               30                     156
               36                     130
               39                     120
               40                     117
               45                     104
               52                      90
               60                      78
               65                      72
       TYPE 1 TO STOP; 2 TO CONTINUE ? 1
```

EXERCISE

1. Find all bases for which 21 is a pseudoprime.
2. Prove that no integer of the form n = 3p (with p > 3 prime) can be pseudoprime to the base 2, 5 or 7.
3. prove that 91 is the smallest pseudoprime to the base 3.
4. Prove that 341 is the smallest pseudoprime to the base 2.
5. Let n = pq be a product of two distinct primes. Let d = gcd(p−1, q−1) prove that n is a pseudoprime to the base b if and only of $b^d \equiv 1$ (mod n).

6. Prove that there are infinitely many pseudoprime to the base b for b = 2, 3, 5.

7. Show by pseudoprime primality test that $2047 = 2^{11} - 1$ is composite.

8. Show that the following are Carmichael numbers.

   6601 , 29341, 172081 , 2465.

9. Show that 65 is a strong pseudoprime to the base 8 and to the base 18, but not to the base 14.

10. Factor n = 8051 using rho method with $f(x) = x^2 + 1$, $x_0 = 1$.

11. Factor 2701 with $f(x) = x^3 = x + 1$, $x_0 = 1$.

12. Use Fermat factorization to factor : *(a)* 809009 *(b)* 8633 *(d)* 4601.

13. Prove that all Carmichael numbers are odd.

14. Verify that 2047 is the smallest strong pseudoprime to the base 2.

15. Show that 2047 is a composite number using strong pseudoprime primality test.

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ANSWER TO EXERCISES

CHAPTER II.

1. (i) 2592 (ii) 22400 (iii) 1800 (iv) 320204 (v) 312 (vi) 720 (viii)1152

CHAPTER III.

1. 4 and 6

5. 7

6. 143

7. 89

8. (a) No solution

   (b) $x = 45,94 \pmod{98}$

   (c) $x \equiv 16, 59, 102, 145, 188, 231$ and $274 \pmod{301}$

   (d) $x = 6, 13$ and $20 \pmod{21}$

9. (a) $x \equiv 4944 \pmod{9889}$ (b) $x \equiv 785 \pmod{1122}$ (c) $x \equiv 52 \pmod{105}$

11. 13

16. $x \equiv 3, 4, 10, 39 \pmod{7}$ are only solutions of the congruence

19. 15

24. $3^2.7.13.7^2$.

25. 82, 43

26. 23

CHAPTER IV. NIL

CHAPTER V. NIL

CHAPTER VI.

1. $\text{ind}_2\, 5 = 9$, $\text{ind}_6\, 5 = 9$, $\text{ind}_7\, 5 = 3$, $\text{ind}_{11}\, 5 = 3$.

2. 14

4. $x \equiv 7, 8$ and $11 \pmod{13}$.

5. (i) $x = 7 + 11\,t$, $t = \pm 1, \pm 2, \ldots$

   (ii)$x = 22 + 23\,t$, $t = 0, \pm 1, \pm 2, \ldots$

   (iii) $x = 7, 29, 57, \ldots, -15, -37$.

6. Solvable.

7. $b \equiv 1, 3, 9 (\text{mod } 13)$

8. (a) $x \equiv 7(\text{mod } 11)$

   (b) $x \equiv 5, 6(\text{mod } 11)$

   (c) No solution .

10. None

CHAPTER VII.

2 (i) –1, (ii) –1, (iii) –1 (iv) 1 (v) +1 (vi) +1.

5. $x \equiv 1,3,4,9,10,12 \pmod{13}$

8. (a) solvable (b) Not solvable (c) solvable 10. (a) –1, (b) –1.

   (c) 1

CHAPTER VIII.

1. 8, 13

10. 8 3, 97

11. 37. 73

12. (a) 823. 983

    (b) 89.97

    (c) 43.107.

APPENDIX

THE CHRONOLOGY OF PRIME NUMBER RECORDS

The Largest Known Prime Records.

On January 27, 1998, the team of Roland Clarkson, George Woltman, Scott Kurowskii et al discovered a new prime record: $2^{3021377} - 1$. In June 1999 largest known prime $2^{6972593} - 1$ having 2,098,960 digits in found by Nayan Hajratwala.

Date	Prime	Digits	Who
Jan 1998	$2^{3021377} - 1$	909526	Clarkson, Woltman, Kurowski, et al. (GIMPS)
Aug 1997	$2^{2976221} - 1$	895932	Spence, Woltman, et al. (GIMPS)
Nov 1996	$2^{1398269} - 1$	420921	Armengaud, Woltman, et al (GIMPS)
Sep 1996	$2^{1257787} - 1$	378632	Slowinski and Gage
Jan 1994	$2^{859433} - 1$	258716	Slowinski and Gage
Feb 1992	$2^{756839} - 1$	227832	Slowinski and Gage
Aug 1989	$391581 \cdot 2^{216193} - 1$	65087	Brown. Noll, Parady, Smith G, Smith J and Zarantonello
Sep 1985	$2^{216091} - 1$	65050	Slowinski
1983	$2^{132049} - 1$	39751	Slowinski
1982	$2^{86243} - 1$	25962	Slowinski
1979	$2^{44497} - 1$	13395	Nelson and Slowinski

Feb 1979	$2^{23209}-1$	6987	Noll
Oct 1978	$2^{21701}-1$	6533	Nickel and Noll
1971	$2^{19937}-1$	6002	Tuckerman
1963	$2^{11213}-1$	3376	Gillies
1961	$2^{4423}-1$	1332	Hurwitz
1957	$2^{3217}-1$	969	Riesel
1952	$2^{2281}-1$	687	Robinson
Jan 1952	$2^{607}-1$	183	Robinson
Jul 1951	$180\cdot(2^{127}-1)^2+1$	79	Miller and Wheeler
1951	$(2^{148}+1)/17$	44	Ferrier
1876	$2^{127}-1$	39	Lucas
1867	$(2^{53}+1)/(3.107)$	14	Landry
1851	999999000001	12	Looff
1771	$2^{31}-1$	10	Euler

THE TWIN PRIME RECORDS

On August 31, 1998, Ray Ballinger has found a 11751 digits twin primes record with "Proth" written by Yves Gallot: $835335 \cdot 2^{39014} \pm 1$.

Date	Prime	Digits	Who
Aug 1998	$835335 \cdot 2^{39014} \pm 1$	11751	Ballinger and Gallot
Nov 1995	$242206083 \cdot 2^{38880} \pm 1$	11713	Indlekofer and Jarai
Oct 1995	$570918348 \cdot 2^{5120} \pm 1$	5129	Dubner
Nov 1994	$697053813 \cdot 2^{16352} \pm 1$	4932	Indlekofer and Jarai
1993	$4655478828 \cdot 10^{5120} \pm 1$	3439	Dubner
1989	$1706595 \cdot 2^{11235} \pm 1$	3389	Brown. Noll, Parady, Smith G, Smith J and Zarantonello

THE MERSENNE PRIME RECORDS

Mersens primes are of the form $2^p - 1$.

On January 27, 1998, the team of Roland Clarkson, George Woltman, Scott Kurowskii et al discovered a new prime record: $2^{3021377} - 1$.

Date	Prime	Digits	Who
Jan 1998	$2^{3021377} - 1$	909526	Clarkson, Woltman, Kurowski, et al. (GIMPS)
Aug 1997	$2^{2976221} - 1$	895932	Spence, Woltman, et al. (GIMPS)
Nov 1996	$2^{1398269} - 1$	420921	Armengaud, Woltman, et al (GIMPS)
Sep 1996	$2^{1257787} - 1$	378632	Slowinski and Gage
Jan 1994	$2^{859433} - 1$	258716	Slowinski and Gage
Feb 1992	$2^{756839} - 1$	227832	Slowinski and Gage
Aug 1989	$391581 \cdot 2^{216193} - 1$	65087	Brown, Noll, Parady, Smith G, Smith J and Zarantonello
Sep 1985	$2^{216091} - 1$	65050	Slowinski
1983	$2^{132049} - 1$	39751	Slowinski
1982	$2^{86243} - 1$	25962	Slowinski
1979	$2^{44497} - 1$	13395	Nelson and Slowinski
Feb 1979	$2^{23209} - 1$	6987	Noll
Oct 1978	$2^{21701} - 1$	6533	Nickel and Noll
1971	$2^{19937} - 1$	6002	Tuckerman
1963	$2^{11213} - 1$	3376	Gillies
1961	$2^{4423} - 1$	1332	Hurwitz
1957	$2^{3217} - 1$	969	Riesel
1952	$2^{2281} - 1$	687	Robinson
Jan 1952	$2^{607} - 1$	183	Robinson
1876	$2^{127} - 1$	39	Lucas
771	$2^{31} - 1$	10	Euler

The 38th Mersenne prime is $2^{6972593}$ -1 is found in June 1998

Number of the form n! ± 1 are called factorial primes.

Date	Prime	Digits	Who
1998	6917! - 1	23560	Caldwell and Gallot
1993	3610! – 1	11277	Caldwell
1992	3507! – 1	10912	Caldwell
1992	1963! – 1	5614	Caldwell and Dubner
1984	1477! – 1	4042	Dubner
1983	872! – 1	2188	Dubner
1981	469! – 1	1051	Buhler, Crandall and Penk

Primorial Primes are of the form 2.3.5.. . . p+1 .

Date	Prime	Digits	Who
1993	24029# +1	10387	Caldwell
1989	18523# +1	8002	Dubnerl
1987	13649# +1	5862	Dubner
1986	11549# +1	4951	Dubner
1984	4787# +1	2038	Dubner
1982	2657# +1	1115	Buhler, Crandall and Penk

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